Topology II Fall 2005 Graduate Center CUNY

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October 24, 2005
Chapter 1

Basic Concepts

We begin with some fundamental concepts that are ubiquitous in homotopy theory.

1.1 Homotopy Groups

Definition 1.1.1. Let $X$ be a space with basepoint $x_0$. Define $\pi_n(X, x_0) = [S^n, e_0; X, x_0]$, the set of homotopy equivalence classes of pointed maps of the $n$-sphere into $X$. If the base point is implicit we will often just write $\pi_n(X) = [S^n, X]$ where it is understood that all maps and homotopies are basepoint preserving.

For $n = 0$ $\pi_0(X)$ is merely a set, but for $n \geq 1$, $\pi_n(X)$ is a group, called the $n$th homotopy group of $X$. The group structure is defined as follows. Let $\mu : S^n \to S^n \vee S^n$ be the map that pinches the equator of $S^n$ to a point. For any space $X$, let $\rho : X \vee X \to X$ be the fold map which maps each copy of $X$ in the wedge homeomorphically to $X$. For two maps $f, g : S^n \to X$ representing elements $[f]$ and $[g]$ in $\pi_n(X)$, define $[f] + [g]$ to be the homotopy equivalence class represented by

$$S^n \xrightarrow{\mu} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\rho} X.$$ 

We leave it as an exercise to verify that this operation is well defined and does indeed make $\pi_n(X)$ into a group. If $f : X \to Y$ then composition yields an induced homomorphism $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ allowing us to view $\pi_n$ as a functor from $T_*$, the category of pointed spaces, to $\mathcal{G}$, the category of groups. It is built into the definition that homotopic maps induce the same
the homomorphism, so we can also think of $\pi_n$ as functor on $T'_*$, the pointed homotopy category of spaces. The reader should also verify that for $n = 0$, $\pi_0(X)$ is just the set of path components of $X$, and for $n = 1$, $\pi_1(X)$ is just the fundamental group of $X$. We also have the following.

**Proposition 1.1.2.** For all $n \geq 2$ and all spaces $X$, $\pi_n(X)$ is abelian.

Shortly after homotopy groups were defined, Proposition 1.1.2 led researchers to believe that higher homotopy groups were probably more or less the same thing as singular homology groups. Nothing could be further from the truth.

**Proof.** We can think of $\pi_n(X)$ as $[I^n, \partial I^n; X, x_0]$, homotopy classes of maps from the unit cube into $X$ which map the boundary of the cube to the base point. With this representation the sum $[f] + [g]$ is represented by

$$(f + g)(t_1, \ldots, t_n) = \begin{cases} f(t_1, \ldots, 2t_n) & \text{if } 0 \leq t_n \leq \frac{1}{2}, \\ g(t_1, \ldots, 2t_n - 1) & \text{if } \frac{1}{2} \leq t_n \leq 1. \end{cases}$$

This is evidently homotopic to

$$h_1(t_1, \ldots, t_n) = \begin{cases} f(t_1, \ldots, 2t_{n-1}, 2t_n) & \text{if } 0 \leq t_{n-1}, t_n \leq \frac{1}{2}, \\ x_0 & \text{if } 0 \leq t_{n-1} \leq \frac{1}{2} \leq t_n \leq 1, \\ x_0 & \text{if } 0 \leq t_n \leq \frac{1}{2} \leq t_{n-1} \leq 1, \\ g(t_1, \ldots, 2t_{n-1} - 1, 2t_n - 1) & \text{if } \frac{1}{2} \leq t_{n-1}, t_n \leq 1. \end{cases}$$

The reader should draw a picture of a square to see what is going on. The map $h_1$ is obviously homotopic to

$$h_2(t_1, \ldots, t_n) = \begin{cases} f(t_1, \ldots, 2t_{n-1}, t_n) & \text{if } 0 \leq t_{n-1} \leq \frac{1}{2}, \\ g(t_1, \ldots, 2t_{n-1} - 1, t_n) & \text{if } \frac{1}{2} \leq t_{n-1} \leq 1. \end{cases}$$

which is then seen to be homotopic to $g + f$ by more of the same. \qed
As the reader knows from a first course in algebraic topology, it takes a bit of work to define and establish the basic properties of singular homology groups. However once this is done they can be computed for a wide variety of spaces by some fairly algorithmic procedures and then used for various applications. Homotopy groups are somewhat the opposite: the definition and establishment of some basic properties are quick and elementary, but homotopy groups are notoriously difficult to compute. That they contain so much geometrical information makes the endeavor worthwhile.

We finish this section by indicating that one can also define relative homotopy groups.

Definition 1.1.3. Given a pair of spaces \((X, A)\) with basepoint \(x_0 \in A\) define \(\pi_n(X, A, x_0) = [D^n, S^{n-1}, e_0; X, A, x_0]\), for \(n \geq 1\). If \(n = 1\), this is merely a set; if \(n \geq 2\) we can define a group structure analogously to Definition 1.1.1; and if \(n \geq 3\) this is an abelian group.

The proof of the following proposition is left to the reader. We suppress the base point from the notation and just write \(\pi_n(X, A)\) but it is understood that all maps and homotopies are basepoint preserving.

Proposition 1.1.4. 1. \(\pi_n(X, \{x_0\}) \cong \pi_n(X)\).

2. \(\pi_n(X, X) = 0\) for all \(n \geq 1\).

3. For each pair \((X, A)\) there is a LES

\[\cdots \rightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{p_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \rightarrow \cdots\]

In the third part \(i_*\) and \(p_*\) are induced by the inclusions, and \(\partial\) is defined as follows: given \(f : (D^n, S^{n-1}) \rightarrow (X, A)\) representing \([f] \in \pi_n(X, A)\), \(\partial([f])\) is represented by \(f|_{S^{n-1}}\).

The following simple characterization of the zero element in \(\pi_n(X, A)\) will be useful later.

Definition 1.1.5. The pair \((X, A)\) is called \(n\)-connected if \(\pi_k(X, A) = 0\) for all \(k \leq n\), and \(A\) meets every path component of \(X\). Note that by Proposition 1.1.4 this is equivalent to saying that \(i_* : \pi_k(A) \rightarrow \pi_k(X)\) is an isomorphism for all \(k < n\) and is surjective for \(k = n\).
Proposition 1.1.6. The pair $(X, A)$ is $n$-connected if and only if for every commutative diagram of spaces, $k \leq n$,

$$
\begin{array}{ccc}
S^{k-1} & \longrightarrow & A \\
\downarrow & & \downarrow i \\
D^k & \overset{f}{\longrightarrow} & X
\end{array}
$$

there exists a map $\tilde{f} : D^k \to A$ extending $f|_{S^{k-1}}$ with $i \circ \tilde{f} \simeq f|_{\text{rel}(S^{k-1})}$.

Proof. Suppose we have an element in $\pi_k(X, A)$ represented by a map $f : (D^k, S^{k-1}) \to (X, A)$ and $\tilde{f}$ exists. Then $[f]$ is also represented by $\tilde{f}$ but $[\tilde{f}] \in \text{im} \pi_k(A, A) = 0$ so $[f] = 0$.

Conversely, suppose $[f] = 0$. Let $H : (D^k, S^{k-1}) \times I \to X$ be a null homotopy. Define $\tilde{H} : (D^k, S^{k-1}) \times I \to A$ by

$$
\tilde{H}(x, t) = \begin{cases} 
H\left(\frac{2x}{2-t}, t\right) & 0 \leq ||x|| \leq 1 - t/2 \\
H\left(||x||, 2 - 2||x||\right) & 1 - t/2 \leq ||x|| \leq 1.
\end{cases}
$$

Then $\tilde{H}_0 = f$, $\tilde{H}(x, t) = f(x)$ for all $x \in S^{k-1}$, and $\tilde{H}_1(D^k) \subset A$, so $\tilde{H}_1$ satisfies the conclusion. \qed

Exercise 1.1.7. Generalize the preceding proposition as follows: If $(X, A)$ is relative a CW complex of dimension $\leq n$, (i.e. all cells in $X$ not in $A$ have dim $\leq n$), and $h : Y \to Z$ is an $n$-equivalence, and suppose given maps which make the following diagram commute:

$$
\begin{array}{ccc}
A & \overset{g}{\longrightarrow} & Y \\
\downarrow i & & \downarrow h \\
X & \overset{f}{\longrightarrow} & Z
\end{array}
$$

Then there exists a map $\tilde{g} : X \to Y$ which is an extension of $g$ and satisfies $h \circ \tilde{g} \simeq f|_{\text{rel}A}$. (Hint: Use induction over the cells of $X - A$ to reduce the statement to the case where $(X, A) = (D^k, S^{k-1})$. Now replace the arbitrary $n$-equivalence $h$ by a homotopic $n$-equivalence which is an inclusion (the so-called mapping cylinder construction) and then apply 1.1.6).
1.2 Fibrations

Definition 1.2.1. A map $E \xrightarrow{p} B$ is said to have the homotopy lifting property (HLP) with respect to the space $X$ if whenever there is a homotopy $H : X \times I \to B$ together with a lifting $f : X \to E$ of $H_0$, then there is a lifting of $H$ to $\tilde{H} : X \times I \to E$ with $\tilde{H}_0 = f$.

If $E \xrightarrow{p} B$ has the HLP with respect to all spaces $X$ then $p$ is called a fibration.

One consequence of this definition is that the property of having a lifting depends on the homotopy class of the map, i.e. if $g_1, g_2 : X \to B$ are homotopic maps, then $g_1$ lifts to $E$ if and only if $g_2$ does.

We can state the definition diagrammatically as follows: Given a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & E \\
\downarrow{i_0} & & \downarrow{p} \\
X \times I & \xrightarrow{H} & B
\end{array}
\]

there exists a map $\tilde{H}$ making the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & E \\
\downarrow{i_0} & & \downarrow{p} \\
X \times I & \xrightarrow{H} & B \\
\end{array}
\]

\[\text{(1.2.2)}\]

We will often express this sort of statement with a single commutative diagram where the convention is that when the solid arrows are hypothesized, the conclusion is the existence of the dotted arrow(s).

One of the most important properties of a fibration is that it induces a LES in homotopy groups.

Definition 1.2.3. Suppose $p : E \to B$ is a fibration. For a base point $b_0 \in B$ let $F = p^{-1}(b_0)$. The space $F$ is called the fiber of $p$ over $b_0$, and the sequence

\[F \xrightarrow{i} E \xrightarrow{p} B\]
is called a fiber sequence.

**Proposition 1.2.4.** Given a fiber sequence as above, there is of LES in homotopy groups:

\[ \cdots \to \pi_n(F, f_0) \overset{i_*}{\to} \pi_n(E, e_0) \overset{p_*}{\to} \pi_n(B, b_0) \overset{\partial}{\to} \pi_{n-1}(F, f_0) \to \cdots \]

**Proof.** We will indicate a portion of the proof and leave the rest as an exercise. First note that since \( p \circ i \) is constant by definition, \( \text{im} \, i_* \subset \ker \, p_* \). Suppose \([\alpha] \in \ker \, p_*\). This means that we have a map \( \alpha : S^n \to E \) and \( p \circ \alpha \) is null homotopic. Let \( H \) be a null homotopy. We have \( H_0 = p \circ \alpha \) and \( H_1 = b_0 \). By the HLP there exists \( \tilde{H} \), a lifting. Thus \( \tilde{H_0} = \alpha \) and \( \tilde{H_1} = b_0 \), i.e. the image of \( \tilde{H_1} \) lies in \( F \). This means that \([\alpha] \in \text{im} \, i_* \), so \( \text{im} \, i_* = \ker \, p_* \).

We will also define the boundary homomorphism \( \partial \). Let \([\alpha] \in \pi_n(B, b_0)\) and represent it by a map \( \alpha : (I^n, \partial I^n) \to (B, b_0) \). Since \( I^n \) is contractible, \( \alpha : I^n \to B \) is null-homotopic (not necessarily relative to \( \partial I^n \)). Let \( G : I^n \times I \to B \) be a null homotopy. Then \( G|_{\partial I^n \times I} \) is also a null homotopy, which we will call \( H \). Since \( H_1 \) is constant, it has a lifting to the constant map \( \partial I^n \to e_0 \) and by the HLP there is a lifting of \( H \) to a homotopy \( \tilde{H} : \partial I^n \times I \to E \). We have \( \tilde{H}_1 = e_0 \) and \( \tilde{H}_0 : \partial I^n \to p^{-1}(B) = F \). We may take this last map to be the definition \( \partial([\alpha]) \).

**Exercise 1.2.5.** Prove the rest of Proposition 1.2.4.

The following example is very useful.

**Definition 1.2.6.** Let \( X \) be a space with basepoint \( x_0 \). Let \( P \times X = (X, x_0)^{(I, 0)} \), the space of paths in \( X \) starting at the point \( x_0 \), with the compact-open topology. Define a map \( p : P \times X \to X \) by \( p(\omega) = \omega(1) \). Notice that \( p^{-1}(x_0) \) is just \( \Omega X = (X, x_0, x_0)^{(I, 0, 1)} \), the space of loops in \( X \) starting and ending at \( x_0 \).

**Proposition 1.2.7.** \( p : P \times X \to X \) is a fibration with fiber \( \Omega X \).

**Proof.** Suppose we’re given

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & PX \\
\downarrow{ia} & & \downarrow{p} \\
Y \times I & \xrightarrow{H} & X
\end{array}
\]
in which the vertical map on the left is defined by \( i_0(y) = (y, 0) \). Define \( \tilde{H} : Y \times I \to PX \) by

\[
\tilde{H}(y, t)(s) = \begin{cases} 
  f(y)(s(t + 1)) & \text{if } 0 \leq s \leq \frac{1}{t + 1}, \\
  H(y, s(t + 1) - 1) & \text{if } \frac{1}{t + 1} \leq s \leq 1.
\end{cases}
\]

to get

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & PX \\
\downarrow{i_0} & & \downarrow{p} \\
Y \times I & \xrightarrow{H} & X
\end{array}
\]

Proposition 1.2.8. \( PX \) is contractible for all \( X \).

Proof. Define \( H : PX \times I \to PX \) by \( H(\omega, t)(s) = \omega(s(1 - t)) \). Then \( H(\omega, 0) = \omega \) and \( H(\omega, 1) = \omega(0) = x_0 \).

The fiber sequence \( \Omega X \to PX \to X \) gives rise to a LES

\[
\cdots \to \pi_n(\Omega X) \to \pi_n(PX) \to \pi_n(X) \xrightarrow{\partial} \pi_{n-1}(\Omega X) \to \cdots
\]

and since \( PX \) is contractible we have \( \pi_n(PX) = 0 \) for all \( n \). This gives us

Corollary 1.2.9. \( \pi_{n-1}(\Omega X) \cong \pi_n(X) \) for all \( n \).

This can easily be proved directly from the definition of \( \pi_n \) and \( \Omega \).

Now we come to an an important class of examples of fibrations called fiber bundles.

Definition 1.2.10. A fiber bundle \( \xi \) is a quadruple \((E, p, B, F)\) where \( p : E \to B \) is a map and \( B \) has an open covering \( \{U_\lambda\}_{\lambda \in \Lambda} \) such that for each \( \lambda \) there is a homeomorphism

\[
\phi_\lambda : U_\lambda \times F \to p^{-1}(U_\lambda)
\]

satisfying \( p(\phi_\lambda(b, f)) = b \).
Intuitively, the idea is that $E$ locally looks like a product, i.e. each point $b \in B$ has a neighborhood over which $p$ is a trivial fibration. Sometimes a fiber bundle is described as a 'twisted product'. We call $B$ the base space, $E$ the total space, and $F$ the fiber.

There is a tradition in notes of this sort to omit the proof of the following fact, presumably because the proof is a little lengthy and may be felt to detract from the flow of ideas. These notes will not be an exception. A proof can be found in Steenrod.

**Theorem 1.2.11.** Let $\xi = (E, p, B, F)$ be a fiber bundle over a paracompact base space $B$. Then $p$ is a fibration.

**Example 1.2.12.** Let $p : \tilde{X} \to X$ be a covering space. This is just a fiber bundle with a discrete fiber. Since $\pi_n$ of a discrete space is zero for $n > 0$, the LES of homotopy groups tells us that $\pi_n(\tilde{X}) \cong \pi_n(X)$ for $n \geq 2$. For instance, the covering space $\mathbb{Z} \to \mathbb{R} \to S^1$ tells us that $\pi_n(S^1) = 0$ for $n \geq 2$.

**Example 1.2.13.** Let $O(n)$ be the orthogonal group, i.e. real $n \times n$ matrices with determinant equal to $\pm 1$. $O(n)$ acts on $\mathbb{R}^n$ on the left by matrix multiplication, thinking of elements in $\mathbb{R}^n$ as $n \times 1$ column vectors. Let $e_1 = (1, 0, \ldots, 0)$ be the first standard basis vector (thought of as a column vector!), and define a map $p : O(n) \to S^{n-1}$ by $p(A) = Ae_1$, $A \in O(n)$. Think of $e_1$ as the base point of $S^{n-1}$ and observe that $F = p^{-1}(e_1)$ consists of orthogonal matrices that fix the one-dimensional subspace spanned by $e_1$, hence $F$ can be identified with $O(n-1)$. It is readily verified that $p : O(n) \to S^{n-1}$ is a fiber bundle with fiber $O(n-1)$.

**Example 1.2.14.** There is an action of the $S^1$ on $S^{2n-1}$ obtained by thinking of $S^{2n-1}$ as $\{(z_1, \ldots, z_n) \in \mathbb{C}^n| |z_1|^2 + \ldots + |z_n|^2 = 1\}$. Then for $\lambda \in S^1$ we have $\lambda(z_1, \ldots, z_n) = (\lambda z_1, \ldots, \lambda z_n)$. Taking the quotient space of $S^{2n-1}$ modulo the relation $(z_1, \ldots, z_n) \sim (w_1, \ldots, w_n)$ if $(z_1, \ldots, z_n) = (\lambda w_1, \ldots, \lambda w_n)$ (the 'orbit space' of the action) we get $p : S^{2n-1} \to \mathbb{C}P^{n-1}$. The inverse image of each point in $\mathbb{C}P^{n-1}$ is $S^1$ and this is in fact a fiber bundle over $\mathbb{C}P^{n-1}$ with fiber $S^1$, called the Hopf bundle.

When $n = 2$ we get $S^1 \to S^3 \to S^2$. The LES in homotopy groups, see Example 1.2.12, tells us that $\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$. We know that $\pi_3(S^3) = \mathbb{Z}$ (from the Hurewicz Theorem for example), and so we get $\pi_3(S^2) = \mathbb{Z}$, which is Hopf’s example of a non-trivial homotopy group above the dimension of a (very nice) space.
1.3 Cofibrations

Let $A \overset{i}{\hookrightarrow} X$ be the inclusion of a closed subspace.

**Definition 1.3.1.** The inclusion $i$ is said to have the homotopy extension property (HEP) with respect to a space $Y$ if whenever there is a homotopy $H : A \times I \rightarrow Y$ and an extension $f : X \to Y$ of $H_0$, then there is an extension of $H$ to a homotopy $\tilde{H} : X \times I \rightarrow Y$ with $\tilde{H}_0 = f$.

We can express this condition diagrammatically by

$$
\begin{array}{ccc}
X \times \{0\} \cup A \times I & \xrightarrow{f \cup H} & Y \\
\downarrow & & \downarrow \\
X \times I & \xrightarrow{\tilde{H}} & Y
\end{array}
$$

If $i$ has the HEP with respect to all spaces $Y$ then $i$ is called a cofibration. The quotient space $X/A$ is called the cofiber of $i$ and

$$A \overset{i}{\hookrightarrow} X \to X/A$$

is called a cofiber sequence.

One consequence of this definition is that the property of having an extension depends on the homotopy class of the map, i.e. if $g_1, g_2 : A \to Y$ are homotopic maps, then $g_1$ extends over $X$ if and only if $g_2$ does.

There is an informal duality between cofibrations and fibrations which can be seen when we represent the definition with a slightly different diagram. Bear in mind that a map from $Z \times I$ to $Y$ is the same thing as a map from $Z$ to $Y^I$. The definition of a cofibration is represented by the following diagram

$$
\begin{array}{ccc}
Y & \xleftarrow{f} & X \\
\downarrow{p_0} & & \downarrow{i} \\
Y^I & \xleftarrow{H} & A
\end{array}
$$

and this diagram is 'dual' to diagram (1.2.2) in the sense that the arrows are reversed and $- \times I$ is replaced by $-^I$ (the names of the spaces have been changed according to convention).

One of the basic properties of a cofiber sequence is that it gives a LES in cohomology.
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Proposition 1.3.3. Given a cofiber sequence \( A \rightarrow X \rightarrow X/A \), there is of LES in singular cohomology groups (with coefficients in any abelian group):

\[
\cdots \rightarrow H^n(X/A) \rightarrow H^n(X) \rightarrow H^n(A) \overset{\partial}{\rightarrow} H^{n+1}(X/A) \rightarrow \cdots
\]

Lemma 1.3.4. If \( A \hookrightarrow X \) is a cofibration and \( A \) is contractible then the projection \( p : (X, A) \rightarrow (X/A, *) \) is a homotopy equivalence.

Proof. See Switzer, page 75. \( \square \)

Proof of Proposition 1.3.3. If \( A \hookrightarrow X \) is a cofibration, it is easy to check that \( CA \hookrightarrow X \cup CA \) is also a cofibration. Since \( CA \) is contractible, by Lemma 1.3.4 the projection map \( p : (X \cup CA, CA) \rightarrow (X \cup CA/CA, *) \) is a homotopy equivalence. It is easy to check that \( (X/A, *) \rightarrow (X \cup CA/CA, *) \) is a homeomorphism. Thus \( H^n(X/A, *) \cong H^n(X \cup CA/CA, *) \cong H^n(X \cup CA, CA) \). A straightforward application of the Excision Theorem shows that \( H^n(X \cup CA, CA) \cong H^n(X, A) \) and the conclusion follows from the LES of the pair \( (X, A) \). \( \square \)

The use of the mapping cone in the above proof suggests a trick whereby we can pretend that any map is a cofibration (even if it isn’t injective). First note that by the above:

Lemma 1.3.5. If \( A \hookrightarrow X \) is a cofibration, then \( X \cup CA \rightarrow X/A \) is a homotopy equivalence.

Definition 1.3.6. If \( Y \xrightarrow{f} X \) is any map of spaces, define the mapping cylinder, \( X \cup_f (Y \times I) \) to be the quotient space of the disjoint union \( X \bigsqcup (Y \times I) \) by making the identification \((y, 0) \sim f(y)\) for each \( y \in Y \). Let \( i : Y \rightarrow X \cup_f (Y \times I) \) be defined by \( i(y) = (y, 1) \).

With the above definition, the following proposition is easy to verify.

Proposition 1.3.7. The map \( i \) is a cofibration with cofiber \( X \cup_f CY \). The inclusion \( X \hookrightarrow X \cup_f (Y \times I) \) is a homotopy equivalence and so we get a LES

\[
\cdots \rightarrow H^n(X \cup_f CY) \rightarrow H^n(X) \xrightarrow{f^\ast} H^n(Y) \overset{\partial}{\rightarrow} H^{n+1}(X \cup_f CY) \rightarrow \cdots
\]

The mapping cone of \( f \) is sometimes called the homotopy cofiber of \( f \) and the sequence \( Y \xrightarrow{f} X \rightarrow X \cup CY \) is sometimes called a homotopy cofiber sequence, or a cofiber sequence in the homotopy category.
Exercise 1.3.8. Starting with any \( f : Y \to X \), Show that the mapping cone of \( X \to X \cup_f CY \) is homotopy equivalent to \( \Sigma Y \), the mapping cone of \( X \cup_f CY \to \Sigma Y \) is homotopy equivalent to \( \Sigma X \), etc.

Exercise 1.3.9. Given a cofibration \( A \hookrightarrow X \) and a fibration \( E \xrightarrow{p} B \) we can consider the following combined homotopy extension lifting property (HELP):

\[
\begin{array}{ccc}
X \times \{0\} \cup A \times I & \rightarrow & E \\
\downarrow & & \downarrow p \\
X \times I & \rightarrow & B \\
\uparrow H & \leftarrow & \\
A & \rightarrow & E \\
\downarrow j & & \downarrow \\
X & \rightarrow & B
\end{array}
\]

Note that if \( A = \emptyset \) then this is just the HLP, and if \( B = * \) then this is just the HEP. Show that this problem can be always be solved under the stated hypothesis.

(Hint: first show that if \( A \hookrightarrow X \) is a deformation retract, then the following problem can always be solved:

\[
\begin{array}{ccc}
A & \rightarrow & E \\
\downarrow & & \downarrow \\
X & \rightarrow & B
\end{array}
\]

Then show that \( X \times \{0\} \cup A \times I \) is a deformation retract of \( X \times I \).)

1.4 Cohomology Theories

In this section we introduce what are often referred to as 'generalized cohomology theories', because they are a generalization of singular cohomology. We will usually just call them 'cohomology theories'.

Let \( T^2 \) denote the category of pairs \((X, A)\) of topological spaces. Let \( R : T^2 \to T^2 \) denote the functor \( R(X, A) = (A, \emptyset) \).

Definition 1.4.1. An (unreduced) cohomology theory is a sequence of contravariant functors \( E^n : T^2 \to \mathcal{A} \), \( n \in \mathbb{Z} \), along with a sequence of natural transformations \( \partial^{n+1} : E^n \circ R \to E^{n+1} \) satisfying the following properties:
1. If \( f, g : (X, A) \to (Y, B) \) are homotopic maps of pairs, then \( E^n(f) = E^n(g) : E^n(Y, B) \to E^n(X, A) \) for all \( n \in \mathbb{Z} \).

2. For each pair \( (X, A) \in T^2 \) there is a LES

\[
\cdots \to E^n(X, A) \to E^n(X, \emptyset) \to E^n(A, \emptyset) \xrightarrow{\partial^{n+1}} E^{n+1}(X, A) \to \cdots
\]

The maps \( E^n(X, A) \to E^n(X, \emptyset) \) and \( E^n(X, \emptyset) \to E^n(A, \emptyset) \) are induced by inclusion.

3. If \((X; A, B)\) is a triad in which \( A \) and \( B \) are closed, \( X = A \cup B \), and the inclusion map \( A \cap B \hookrightarrow A \) is a cofibration, then the inclusion map \((A, A \cap B) \hookrightarrow (X, B)\) induces an isomorphism \( E^n(X, B) \cong E^n(A, A \cap B) \) for all \( n \in \mathbb{Z} \).

Classical singular cohomology, with coefficients in an abelian group \( A \), satisfies the above properties, along with the additional condition that \( H^n(\ast, \emptyset) = 0 \) unless \( n = 0 \) in which case we have \( H^0(\ast, \emptyset) = A \). This last condition is called the 'dimension axiom', and Eilenberg and Steenrod proved that together with the dimension axiom uniquely determine singular cohomology. Thus the seemingly innocuous dimension axiom is the only thing that separates ordinary cohomology from generalized cohomology, the latter concept having a plethora of rich examples which differ significantly from the former.

**Exercise 1.4.2.** Show that the proof of Proposition 1.3.3 works for any cohomology theory.

We have the closely related notion of a reduced cohomology theory. Let \( T_* \) denote the category of topological spaces with non-degenerate base point, i.e. the inclusion \( \{\ast\} \to X \) is a cofibration.

**Definition 1.4.3.** A reduced cohomology theory is a sequence of contravariant functors \( \tilde{E}^n : T_* \to A \) for each \( n \in \mathbb{Z} \), along with a natural transformation \( \sigma^n : \tilde{E}^{n+1} \circ \Sigma \to \tilde{E}^n \) satisfying the following properties:

1. If \( f, g : X \to Y \) are homotopic maps of pointed spaces, then \( \tilde{E}^n(f) = \tilde{E}^n(g) : \tilde{E}^n(Y) \to \tilde{E}^n(X) \) for all \( n \in \mathbb{Z} \).

2. If \( A \xrightarrow{i} X \xrightarrow{p} X/A \) is a cofiber sequence with \( \ast \in A \), then there is an exact sequence:

\[
\tilde{E}^n(X/A) \xrightarrow{\tilde{E}^n(p)} \tilde{E}^n(X) \xrightarrow{\tilde{E}^n(i)} \tilde{E}^n(A).
\]
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3. The map $\sigma^n(X) : \tilde{E}^{n+1}(\Sigma X) \cong \tilde{E}^n(X)$ is an isomorphism for all $X \in \mathcal{T}$ and $n \in \mathbb{Z}$.

One thing we notice is that the exact sequence in part 2. is not long. This is easily remedied by the following:

**Proposition 1.4.4.** Assuming Definition 1.4.3 there is a LES

$$
\cdots \to \tilde{E}^n(X/A) \xrightarrow{\tilde{E}^n(p)} \tilde{E}^n(X) \xrightarrow{\tilde{E}^n(i)} \tilde{E}^n(A) \xrightarrow{\partial} \tilde{E}^{n+1}(X/A) \to \cdots
$$

in which $\partial$ is defined by

$$
\tilde{E}^n(A) \xrightarrow{(\sigma^n)^{-1}} \tilde{E}^{n+1}(\Sigma A) \to \tilde{E}^{n+1}(X \cup CA) \cong \tilde{E}^{n+1}(X/A).
$$

The middle map is induced by the map $X \cup CA \to \Sigma A$ which collapses $X$ to a point, and the last isomorphism is from Lemma 1.3.5.

**Proof.** To prove exactness at $\tilde{E}^n(A)$ notice that by the following diagram, which commutes by the naturality of $\sigma$,

$$
\begin{array}{ccc}
\tilde{E}^n(A) & \xrightarrow{\cong} & \tilde{E}^{n+1}(\Sigma A) \\
\downarrow & & \downarrow \\
\tilde{E}^n(X) & \xrightarrow{\cong} & \tilde{E}^{n+1}(\Sigma X)
\end{array}
$$

it is equivalent to prove exactness of

$$
\tilde{E}^{n+1}(\Sigma X) \to \tilde{E}^{n+1}(\Sigma A) \to \tilde{E}^{n+1}(X \cup CA).
$$

But this follows from part 2 of the definition and the homotopy cofiber sequence

$$
X \cup CA \to \Sigma A \to \Sigma X.
$$

Exactness at $\tilde{E}^{n+1}(X/A)$ is similar: it is equivalent to prove exactness of

$$
\tilde{E}^{n+1}(\Sigma A) \to \tilde{E}^{n+1}(X \cup CA) \to \tilde{E}^{n+1}(X)
$$

which follows from part 2 and the cofiber sequence

$$
X \to X \cup CA \to \Sigma A.
$$

$\square$
It is mainly a technical matter, having to do with issues of base points and such, that we sometimes consider unreduced theories and sometimes reduced theories. Fortunately it is simple to describe the relationship between the two notions.

**Proposition 1.4.5.** Given an unreduced cohomology theory $E$ and a pointed space $X$, there is a corresponding reduced cohomology theory defined by $\tilde{E}^n(X) = E^n(X, x_0)$. Conversely, given a reduced cohomology theory $\tilde{E}$ and a pair $(X, A)$, there is a corresponding unreduced theory defined by $E^n(X, A) = \tilde{E}^n(X/A)$.

We leave the proof as an exercise.

### 1.5 Spectra

Now we turn our attention to the question of how one might obtain a cohomology theory. As it turns out, the construction used to obtain singular cohomology – whereby one starts by considering $S_n(X)$, the free abelian group on the set of all maps of a standard $n$-simplex into a space $X$, together with a boundary homomorphism $S_n(X) \to S_{n-1}(X)$ making this into a chain complex, and then considers the hom dual of this which is a cochain complex, and looks at the cohomology groups of this cochain complex – is rather specific to this case. Other cohomology theories, $K$-theory and complex cobordism for instance, are not constructed in an analogous fashion.

Instead we start with the following simple definition.

**Definition 1.5.1.** A spectrum is a sequence of pointed spaces $Y_0, Y_1, Y_2, \ldots$, together with structure maps $\epsilon_n : \Sigma Y_n \to Y_{n+1}$ for all $n \geq 0$.

**Remark 1.5.2.** Because of the adjointness relation $Z^{\Sigma Y} = (\Omega Z)^Y$ for all spaces $Y, Z$ we could just as well define a spectrum to be a sequence of pointed spaces $Y_0, Y_1, Y_2, \ldots$, together with structure maps $\epsilon'_n : Y_n \to \Omega Y_{n+1}$ for all $n \geq 0$. This would be the same thing.

A spectrum defines a cohomology theory.

**Definition 1.5.3.** Let $X$ be a space, and let $Y = \{Y_0, Y_1, \ldots\}$ be a spectrum. Define a functor $\tilde{E}^n : T'_* \to \mathcal{A}$ by

$$\tilde{E}^n(X) = \lim\{[X, Y_n] \to [\Sigma X, Y_{n+1}] \to [\Sigma^2 X, Y_{n+2}] \to \ldots\}$$
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where these are basepoint preserving homotopy classes of pointed maps, and
the function \([X, Y_n] \to [\Sigma X, Y_{n+1}]\) is induced by \(\epsilon_n\).

Define a natural transformation \(\sigma_n : \tilde{E}^{n+1}(\Sigma X) \to \tilde{E}^n(X)\) by the diagram

\[
\begin{array}{ccccccc}
[X, Y_n] & \rightarrow & [\Sigma X, Y_{n+1}] & \rightarrow & [\Sigma^2 X, Y_{n+2}] & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
[\Sigma X, Y_{n+1}] & \rightarrow & [\Sigma^2 X, Y_{n+2}] & \rightarrow & \cdots
\end{array}
\]

**Proposition 1.5.4.** The functor \(\tilde{E}^n\) defined in this way is a generalized cohomology theory. The spectrum \(Y\) is called a representing spectrum for \(\tilde{E}^n\).

**Proof.** It is built into the definition that \(\tilde{E}^n(X)\) is well defined on the homotopy category, and the natural transformation \(\sigma_n\) is obviously a natural equivalence. Also note that \(\tilde{E}^n(X)\) is an abelian group, since from the second term in the limit onward, we are taking maps out of a double suspension. It remains to check part 2). If \(A \xrightarrow{i} X \xrightarrow{j} X/A\) is a cofiber sequence, and \(Y\) is a space, we need to check that

\[
[X/A, Y] \xrightarrow{j^*} [X, Y] \xrightarrow{i^*} [A, Y]
\]

is exact. That \(\text{im} \ j^* \subset \ker \ i^*\) follows from the fact that \(A \to X \to X/A\) is constant. That \(\ker \ i^* \subset \text{im} \ j^*\) follows immediately from the HEP. The result now follows since direct limits of abelian groups preserve exactness. \(\square\)

**Remark 1.5.5.** There is a result of fundamental importance in algebraic topology due to E. Brown which gives a converse to this. Given a cohomology theory on the homotopy category of CW complexes which satisfies certain reasonable conditions, there exists a representing spectrum. Since we don’t need this theorem right now, I won’t give a precise statement.

So we see that cohomology theories are abundant – given any spectrum we get one. For instance, we could start with any space \(Y\), and define a spectrum, called the suspension spectrum of \(Y\), by \(Y_n = \Sigma^n Y\). The structure map \(\epsilon_n\) is the identity. This gives a cohomology theory.

However there’s a catch. If we are to have any hope of computing \(\tilde{E}^n(X)\) for various spaces \(X\) we should at least be able to start by computing the
coefficients of the theory $\tilde{E}^n(S^0)$ (equivalently $E^n(pt)$). But we see from Definition 1.5.3 that this amounts to computing homotopy groups of the spaces $Y_n$, or at least direct limits of homotopy groups of the spaces $Y_n$. So if we are to have any hope of making calculations with a cohomology theory, the spaces in the representing spectra had better be of the sort that we can get a handle on their homotopy groups. This rules out, at least for now, examples like the suspension spectrum of a finite complex, as the problem of computing the homotopy groups of such a thing is notoriously difficult and is just the sort of problem that we may wish to approach in the future by using, for instance, generalized cohomology theories.

In the next chapter, we will study fiber bundles, and this will lead us to a construction of the spectrum used to define $K$-theory.