Hovey's paper on Morita theory for Hopf algebroids

Rob Thompson

April 29, 2020

Affine Group Schemes

Let A be an algebra (commutative associative, with unit) over a ground ring k. Then $\operatorname{Spec} A$ is functor from k-algebras to sets where $\operatorname{Spec} A(R) = \operatorname{Hom}_{k-alg}(A, R)$. An *affine group scheme* is such a representable functor from k-algebras to groups. More specifically, let A be a k-algebra with a k-algebra map $A \xrightarrow{\Delta} A \otimes A$ inducing a binary operation on $\operatorname{Spec} A(R)$ as follows:for $f, g \in \operatorname{Hom}(A, R)$ define fg by

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} R \otimes R \xrightarrow{m} R.$$

Then A is called an affine group scheme if this operation makes $\operatorname{Spec} A(R)$ into a group, naturally in R. It can be proved that A represents an *affine group scheme* if and only if A has such a map Δ, as well as maps $A \xrightarrow{c} A$ and $A \xrightarrow{\epsilon} k$ making certain diagram commutes. The map c goes by various name, e.g. conjugation, inversion, etc. and ϵ is usually called a counit. A is called a *Hopf algebra*.

Hopf Algebroids

A *Hopf algebroid* is an affine groupoid scheme. Specifically, it is a pair of k-algebras (A, Γ) where $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ is a functor from k-algebras to groupoids. The functor $\operatorname{Spec} A$ gives the objects and $\operatorname{Spec} \Gamma$ gives the morphisms. Unwinding this definition means that A and Γ are k-algebras with maps:

$A \xrightarrow{\eta_L} \Gamma$	left unit or source,
$A \xrightarrow{\eta_R} \Gamma$	right unit or target,
$\Gamma \xrightarrow{\Delta} \Gamma \otimes_A \Gamma$	coproduct or composition,
$\Gamma \xrightarrow{c} \Gamma$	conjugation or inverse
$\Gamma \xrightarrow{\epsilon} A$	counit or identity.

making various diagrams commute. Note that there are **two** units, a "left" one, and a "right" one, and they will in general be different. The maps Δ and ϵ are A-bimodule maps.

Some authors set it up so that η_L represents the target and η_R represents the source in the groupoid. As in the affine group case, the map $\Gamma \xrightarrow{\Delta} \Gamma \otimes_A \Gamma$ gives the product, defined when the target of the first factor equals the source of the second.
Hopf algebroids arise in algebraic topology, because when E is a ring spectrum representing a generalized homology theory, then the coefficient ring $\pi_* E = E_*$ and the ring of cooperations $\pi_* E \wedge E = E_* E$ usually form a Hopf algebroid, which is frequently not a Hopf algebra.

Example

Brown-Peterson homology, which is used all the time, is a p-local homology represented by a ring spectrum denoted BP with the Hopf algebroid $(BP_*, BP_*BP) = (Z_{(p)}[v_1, v_2, \ldots], BP_*[t_1, t_2, \ldots])$. The map η_L is the evident one given by the inclusion of the coefficients and the map η_R is fantastically complicated (that's where all the action is!).

Change of rings theorems

There are a number theorems in the homological algebra of Hopf algebroids which relate the Ext groups over one Hopf algebroid to those of another. Here is one of the most well known such theorems, which is used all the time. This theorem is known as the Morava Change of Rings Thereom and was discovered by Jack Morava in the 70s. The following version is an adaptation due to Miller and Ravenel and is the lynchpin of the work by Miller-Ravenel-Wilson, as well as forming the basis for the modern 'chromatic' point of view.

In addition to the above described Hopf algebroid for BP, there is a Hopf algebroid $(E(n)_*, E(n)_*E(n))$ where

$$E(n)_* = Z_{(p)}[v_1, v_2, \ldots, v_n, v_n^{-1}]$$

(note that v_n is inverted and the higher v_i's are missing) and

$$E(n)_*E(n) = E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} BP_*BP.$$

Theorem

Suppose we have a BP_*BP-comodule M which is I_n-nil, which means that each element in M is annhilated by some power of the ideal $I_n = (v_1, v_2, \ldots, v_{n-1})$. Denote $\overline{M} = M \otimes_{BP_*} E(n)_*$. Then there is a natural isomorphism

$$\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}M) = \mathrm{Ext}_{E(n)_*E(n)}(E(n)_*, \overline{M}).$$

The basic example of an I_n-nil comodule is $M = BP_*/I_n$. One interpretation of what this theorem is saying is that if you want to compute Ext over BP_*BP of an I_n-nil comodule and you want to work with v_n inverted, then you might as well throw away all the higher v_i's, which will greatly simplify your calculation without affecting the answer.

Example

Let $n = 1$. Then we essentially get the p-local Connor-Floyd theorem because $E(1)$ is p-local complex K-theory.

Sites, Presheafs, and Sheafs

Let's recall the definition of a sheaf. Let $\mathbf{C}$ be a category. A *Grothendieck topology* on $\mathbf{C}$ is a collection of families of morphisms $\{U_i \to U\}$ in $\mathbf{C}$, called *coverings*, such that

1. If $\{U_i \to U\}$ is a covering, and $V \to U$ is a map, then the pullbacks $\{U_i \times_U V \to V\}$ exist and are a covering.
2. If $\{U_i \to U\}$ is a covering, and for each i, if $\{V_{i,j} \to U_i\}$ is a covering, then $\{V_{i,j} \to U\}$ is a covering.
3. For each isomorphism $V \to U$ in $\mathbf{C}$, The singleton family $\{V \to U\}$ is a covering.

A *site* is a pair $(\mathbf{C}, \mathcal{T})$ where $\mathbf{C}$ is a category and $\mathcal{T}$ is a Grothendieck topology on $\mathbf{C}$. A *presheaf* on a site is contravariant functor $\mathcal{F}$ on $\mathbf{C}$ (to sets, groups, etc). (NOTE: the definition of a presheaf does not depend on the particular Grothendieck topology and only depends on the underlying category $\mathbf{C}$.)

A *sheaf* on a site is a presheaf $\mathcal{F}$ such that

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \times_U U_j)$$

is an equalizer.

Examples (of sites)

1. Let X be a topological space and let **C** be the category of open sets of X with the morphisms being inclusion. Let the coverings be the usual open coverings by open sets.
2. Let $\mathbf{C} = \mathbf{Aff} = \mathbf{Rings}^{\mathrm{op}}$, the opposite category of the category of rings. The trivial topology is the one where the coverings of R are just isomorphisms $S \to R$.
3. Let $\mathbf{C} = \mathbf{Aff}$ and now let $\mathcal{T}$ be the flat topology. We'll say what this is as we go along.

Quasi-coherent sheaves and comodules

Let X : **Rings** $\to$ **Sets** be a functor. Suppose **Aff** has a Grothendieck topology $\mathcal{T}$ and think of X as a presheaf on $\mathbf{Aff}_{\mathcal{T}}$. Define a category $\mathbf{Aff}_{\mathcal{T}}/X$ as follows. The objects will be morphisms of presheaves $\operatorname{Spec} R \to X$ and the morphisms will be commuting triangles. It is convenient to observe that $(\mathbf{Aff}_{\mathcal{T}}/X)^{\mathrm{op}}$ is the same as the category $\mathrm{Points}(X)$. This is defined to be the category of pairs (R, x) where R is a ring and $x \in X(R)$. The morphisms in $\mathrm{Points}(X)$ are the ring maps $(R, x) \xrightarrow{f} (S, y)$ such that $X(f)(x) = y$.

The category $\mathbf{Aff}_{\mathcal{T}}/X$ inherits the Grothendieck topology from $\mathbf{Aff}_{\mathcal{T}}$. A cover of (R, x) is a collection $\{(R, x) \to (S_i, x_i)\}$ such that $\{R \to S_i\}$ is a cover of R.

There is a structure presheaf of rings $\mathcal{O} : (\mathbf{Aff}_{\mathcal{T}}/X)^{\mathrm{op}} \to$ **Rings** given by $\mathcal{O}((R, x)) = R$.

Definition

Let X : **Rings** $\to$ **Sets** be a presheaf of sets on $\mathbf{Aff}_{\mathcal{T}}$. Then a *sheaf of modules over* X, is a sheaf of $\mathcal{O}$-modules on $\mathbf{Aff}_{\mathcal{T}}/X$.

Let's unravel the definition. A sheaf of modules over X is a functorial assignment of an R-module M_x to each point (R, x). So a map $(R, x) \xrightarrow{f} (S, y)$ induces a map of R-modules $M_x \xrightarrow{\theta(f)} M_y$, where M_y is an R-module by restriction, and $\theta(1) = 1$ and $\theta(gf) = \theta(g) \circ \theta(f)$. The sheaf condition says that if $\{(R, x) \to (S_i, x_i)\}$ is a cover, then

$$M_x \to \prod_i M_{x_i} \rightrightarrows \prod_{j,k} M_{x_{j,k}}$$

is an equalizer of R-modules. Here $x_{j,k}$ is the image of x in $X(S_j \otimes_R S_k)$.
So we get a category of sheaves over X, which is denoted $\mathbf{Sh}_X^{\mathcal{T}}$.
Now comes an all-important condition.

Definition

Suppose X : **Rings** $\to$ **Sets** is a functor. A *quasi-coherent sheaf M over X* is a sheaf in the trivial topology such that, given a map $(R, x) \to (S, y)$ of points of X, the map $S \otimes_R M_x \to M_y$ is an isomorphism of R-modules.

Demazure and Gabriel's theorem

Theorem (DG 1980)

Let A be an object in **Rings** *and* $\operatorname{Spec} A$: **Rings** $\to$ **Sets** *the functor represented by A. Then the category of A-modules is equivalent to the category of quasi-coherent sheaves over* $\operatorname{Spec} A$*.*

The equivalence is given as follows:

1. A point over $\operatorname{Spec} A$, (R, x), is a ring map $x : A \to R$. Let M be an A-module, and define a sheaf $\tilde{M}$ over $\operatorname{Spec} A$ to have value on (R, x) given by $\tilde{M}_x = R_x \otimes_A M$. Here R_x stands for the bimodule with R acting on the left, A acting on the right by x.
2. The inverse takes the quasi-coherent sheaf N to its value at $1 : A \to A$.

Sheaves over groupoid functors

Definition

Let (X_0, X_1) be a presheaf of groupoids on $\mathbf{Aff}_{\mathcal{T}}$. A *sheaf* over (X_0, X_1) is a sheaf M over X_0 together with an isomorphism $\psi : \mathrm{dom}^* M \to \mathrm{codom}^* M$ of sheaves over X_1 satisfying the following condition: suppose α is a morphism in the groupoid $(X_0(R), X_1(R))$, i.e. (R, α) is a point of X_1. Then $\psi_\alpha : M_{\mathrm{dom}\alpha} \to M_{\mathrm{codom}\alpha}$ is an isomorphism of R-modules, and if α and β are composable morphisms, we require that $\psi_{\alpha\beta} = \psi_\beta \circ \psi_\alpha$.

Thus we get $\mathbf{Sh}^{\mathcal{T}}_{(X_0,X_1)}$, the category of sheaves over (X_0, X_1).

Definition

A *quasi-coherent sheaf over* (X_0, X_1) is a sheaf M over (X_0, X_1) in the trivial topology such that M is quasi-coherent as a sheaf over X_0.

This gives the category $\mathbf{Sh}^{\mathrm{qc}}_{(X_0,X_1)}$.

Now, a Hopf algebroid (A, Γ) is just a pair of rings such that $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ is a presheaf of groupoids on $\mathbf{Aff}_{\mathcal{T}}$ in the trivial topology.
Finally, we can state Hovey's first theorem.

Theorem (Hovey, 2001)

Suppose (A, Γ) is a Hopf algebroid. Then there is an equivalence of categories between Γ-comodules and quasi-coherent sheaves over $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$.
Next Hovey defines an internal equivalence of presheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$.

We'll start by saying what it is in the case where $\mathcal{T}$ happens to be the trivial topology. A map $\Phi : (X_0, X_1) \to (Y_0, Y_1)$ is an *internal equivalence* if and only if $\Phi(R)$ is full, faithful, and essentially surjective for every R. This means that $\Phi(R)$ is an equivalence of groupoids for all R.

In the case where $\mathcal{T}$ is an arbitrary topology we require that Φ be full, faithful, and *sheaf theoretically essentially surjective* for every R, meaning that for every point (R, y) there is a cover $\{R \to S_i\}$ of R in $\mathcal{T}$, such that $y_i = f_i y$ is in the essential image of Φ for every i.

Hovey's second theorem states

Theorem

Suppose $\Phi : (X_0, X_1) \to (Y_0, Y_1)$ is an internal equivalence of presheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$. Then $\Phi^* : \mathbf{Sh}^{\mathcal{T}}_{(Y_0,Y_1)} \to \mathbf{Sh}^{\mathcal{T}}_{(X_0,X_1)}$ is an equivalence of categories.

Finally, we come to what could be considered the main theorem of the paper. First we need a definition.

Definition

The *flat topology* on **Aff** is defined by taking covering to be finite collection of maps $\{R \to S_i\}$ such that each S_i is flat over R, and the product $\prod_i S_i$ is faithfully flat over R.

Theorem

Suppose $\Phi : (X_0, X_1) \to (Y_0, Y_1)$ *is an internal equivalence of presheaves of groupoids on* $\mathbf{Aff}_{\mathcal{T}}$*, where* $\mathcal{T}$ *is the flat topology. Then* $\Phi^* : \mathbf{Sh}^{qc}_{(Y_0,Y_1)} \to \mathbf{Sh}^{qc}_{(X_0,X_1)}$ *is an equivalence of categories.*

Sometimes the flat topology is called the *fpqc* topology, which stands for "fidèlement plat et quasi-compact", which means "faithfully flat and quasi-compact".

Hovey then brings this all back to reality in the fourth theorem of the paper. Here he gives a concrete condition, in the style of Miller-Ravenel-Wilson, which charactorizes internal equivalences in the flat topology.

Theorem

Suppose $f = (f_0, f_1) : (A, \Gamma) \to (B, \Sigma)$ *is a map of Hopf algebroids. Then*
$f^* : (\operatorname{Spec} B, \operatorname{Spec} \Sigma) \to (\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ *is an internal equivalence in the flat topology if and only if*

$$\eta_L \otimes f_1 \otimes \eta_R : B \otimes_A \Gamma \otimes_A B \to \Sigma$$

is an isomorphism and there is a ring map $g : B \otimes_A \Gamma \to C$ *such that* $g(1 \otimes \eta_R)$ *exhibits* C *as a faithfully flat extension of* A*.*

An example

Let $m \geq n$ and consider the diagram of Hopf algebroids

$$\begin{array}{ccc}
(BP_*, BP_*BP) & \longrightarrow & (v_n^{-1}BP_*/I_n, v_n^{-1}BP_*BP/I_n) \\
\downarrow & & \downarrow \\
(E(m)_*, E(m)_*E(m)) & \longrightarrow & (v_n^{-1}E(m)_*/I_n, v_n^{-1}E(m)_*E(m)/I_n)
\end{array}$$

Suppose M is a BP_*-comodule which is I_n-nil and v_n acts bijectively. Then using a Miller-Ravenel argument with the Landweber filtration theorem and a straightforward argument with the cobar complex, one sees that the two vertical maps induce change of rings isomorphisms.

Strickland showed, using a theorem of Lazard's, that the bottom horizontal map satisfies the hypothesis of the Hopkins-Hovey theorem, so the conclusion is that the top map induces an isomorphism, which is the Hovey-Sadofsky Change of Rings Theorem.

In the case where $m = n$, this is just Miller and Ravenel's version of the Morava change of rings theorem.

The Hovey-Sadofsky change of rings theorem is used in turn by Ravenel, to show that the Adams spectral sequence for $E(n)$ converges.