Commutative Algebra, Group Theory, and Meromorphic differential Equations

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Abstract

In this talk I shall look at the problems of local reduction of meromorphic differential equations, individual as well as depending on parameters, from the perspective of commutative algebra and group theory. The issues discussed will include derivation of canonical forms in both settings, and construction of moduli spaces. I shall focus on irregular singular equations.

Classification of singularities

Regular Solutions have at most polynomial growth at the singularities. Example

$$\frac{du}{dz} = \frac{1}{z} A u \quad (A \text{ holomorphic })$$

Irregular Not regular

Global equations in the regular case

$$\frac{du}{dz} = \sum_j \frac{A_j}{z - a_j} u \quad (A_j \text{ holomorphic })$$

Discussion

Regular singular equations go back to Riemann who discovered that they are determined by their monodromy. Since the equations are linear meromorphic, the solutions are well defined as soon as the initial data are given at a point z_0, in any simply connected domain containing z_0 which is free of the singularities of the coefficients. They can thus be analytically continued along any path γ starting and ending at z_0 which does not meet any point of the set S of singularities; moreover the continuation depends only on the homotopy class of the path. The data at z_0 form a $\mathbf{C}^n$ and so we have a linear transformation $M(\gamma) : u \mapsto v$ where v is the value at z_0 of the solution that results from analytic continuation of the initial solution with value u at z_0. The map $M : \gamma \mapsto M(\gamma)$ gives a representation of the fundamental group of $\mathbf{CP}^1 \setminus S$ in $\mathbf{C}^n$. The Riemann–Hilbert problem asks for a determination of the regular singular equation with a given set of singularities and a given monodromy representation. It has a very colorful history.

Fuchs came to his ideas independently of Riemann because Riemann's results were not published till after many years of his death. Fuchs discovered the local criterion in terms of the orders of the singularities of the coefficients for a given point to be a regular singularity. He also made the beautiful discovery that the periods of the Riemann surfaces of an analytic family satisfy equations with regular singularities; nowadays these are called *Picard–Fuchs* differential equations. For the curves of genus 1, this result goes back to Legendre who found expressions for the periods of elliptic functions as hypergeometric functions. Recently Connes and Kreimer have discovered connections of the R–H problem with renormalization in quantum field theory.

For a general survey as well as references to the literature see my paper in *Expositiones Mathematicae,* **9** (1991), 97–188

General features of regular singular equations

> Local theory based on the result that formal solutions converge (Frobenius).
>
> Global classification completely determined by monodromy (Riemann–Hilbert).
>
> Generalized hypergeometric equations.
>
> Regular singular (integrable) systems in several variables (Appell, Picard, Deligne, Deligne–Mostow).

Hypergeometric equations are of the second order and are precisely the equations with three singular points which can be taken as $), 1, \infty$. Special place for equations whose solutions are algebraic (Klein), whose monodromy is discrete (Schwarz in one dimension and Deligne-Mostow in higher dimensions). In a famous and influential monograph Deligne formulated the notion of regular singularity in higher dimensions and solved the analogue of the R–H problem in that case. Locally such equations are of the form

$$\frac{\partial u}{\partial z_i} = A_i u \ (1 \leq i \leq n)$$

and if the A_i are of the form

$$A_i = \frac{B_i}{z_i}, \qquad B_i \text{ holomorphic}$$

then 0 is a regular singularity. For the equations to have a local solution corresponding to any initial condition at 0 the *Frobenius condition of integrability* must be satisfied:

$$F_{ij} := A_{i,j} - A_{j,i} + [A_i, A_j] = 0 \qquad (1 \leq i, j \leq n).$$

The coordinate invariant notion is that of a *connection* (holomorphic), the solutions are the horizontal sections, and the Frobenius condition is simply that the connection has zero curvature, i.e., is *flat.* In dimension 1 this is automatic but not so in higher dimensions. Such equations are harder to find in dimension > 1. Most equations one encounters are the Picard–Fuchs equations. They were first treated in generality in higher dimensions by Griffiths.

Local theory at irregular singularity

$$\frac{du}{dz} = A(z)u, \qquad u = (u_i)_{1\leq i\leq n}, \qquad A = (a_{ij})_{1\leq i,j\leq n}$$

$$v = gu \qquad g \text{ ``a gauge transformation''}$$

$$\frac{dv}{dz} = B(z)u \qquad B = gAg^{-1} + \frac{dg}{dz}g^{-1}$$

Formal no longer same as analytic. Three parts to the problem.

A. Formal canonical forms.

B. Analytic classification for a given formal structure.

C. Moduli.

Discussion

Gauge transformations arise in physics where they come as functions on spacetime with values in a symmetry group G. The idea was that the symmetries of the system could depend on the location of the observer and so could be spacetime-dependent. Algebraically, if G is a Lie or algebraic group defined over $k = \mathbf{R}$ or $\mathbf{C}$, G may be viewed as a functor which assigns to any k–algebra R the group of R–points $G(R)$; in the analytic context the functor assigns to any smooth manifold R the group $G(R)$ of smooth maps from R to G. The groups $G(R)$ are the gauge groups. For $G = GL(n)$ the gauge groups are $GL(n, R)$.

The point of view of reduction theory is that the gauge groups $GL(n, R)$ act on the (connection)matrices $A(R)$ by

$$g, A \longmapsto g[A] = gAg^{-1} + \partial g \cdot g^{-1}$$

for *differential* $\mathbf{C}$–algebras R; here by a differential algebra we mean an algebra R with a derivation ∂ vanishing over $\mathbf{C}$. The important cases are $R = \mathbf{C}z[\frac{1}{z}], \mathbf{C}[[z]][\frac{1}{z}]$, and their algebraic closures. The problem of reduction may be viewed as the problem of classifying the orbits under $GL(n, R)$.

Problem A

There are two aspects to formal reduction theory of a connection matrix $A \in \mathfrak{gl}(n, \mathbf{C}_z)$ where $\mathbf{C}_z = \mathbf{C}[[z]][\frac{1}{z}]$.

1. Ramification.
2. Connection with geometry of orbits in $\mathfrak{gl}(n, \mathbf{C})$, especially nilpotent orbits.

Ramification was discovered by Fabry in 1885. It says that reduction has to be over not $\mathbf{C}_z$ but over its algebraic closure, i.e., the field formal Laurent series in $z^{\frac{1}{b}}$ (no bounds on b). This is quite natural; in linear algebra spectral theory becomes simple only if the field is algebraically closed.

Connection with nilpotent orbit structure discovered by Babbitt–Varadarajan in 1980. The formal canonical forms go back to Hukuhara, Levelt, Turrittin.

Discussion

Any A can be written as $A^0 + fI$ where $\mathrm{tr}(A^0) = 0$ and $f = -\frac{1}{n}\mathrm{tr}(A)$. The reduction is trivial in dimension 1 and the reduction of fI does not interfere with that of A^0. So we may assume that $A \in \mathfrak{sl}(n, \mathbf{C}_z)$. Let

$$A = z^r A_r + z^{r+1} A_{r+1} + \ldots \ (r < -1, A_r \neq 0, \ \mathrm{tr}\ (A) = 0).$$

Then one can *decouple* A along the spectral subspaces of A_r:

$$A = A^{(1)} \oplus \ldots \oplus A^{(m)},$$

and induction on dimension applies. The gauge transformation is in $GL(n, \mathbf{C}[[z]])$ and so does not change the order r.

If A_r is nilpotent this method fails. Special gauge transformations in $z^{\frac{1}{b}}$ for a suitable integer $b > 1$, are now applied to go from A to A' with leading coefficient $A'_{r'}$ $(r < r')$ such that one of the 2 conditions below are satisfied:

(a) $A'_{r'}$ has two distinct eigenvalues

(b) $A'_{r'}$ is nilpotent but is "better ".

Finite termination of the reduction algorithm

Remark. The key point is that if the order r has to be increased, one has to use gauge transformations which have a pole at $z = 0$. Now, analogous to the Cartan decomposition $GL(n, \mathbf{C}) = U(n)DU(n)$ where $U(n)$ is the unitary group and D is the diagonal group with entries that are real and decreasing, we have the *Birkhoff decomposition* $GL(n, \mathbf{C}_z) = KDK$ where $K = GL(n, \mathbf{C}[[z]])$ and D is the diagonal group with entries $z^{d_i}(d_1 \geq \ldots \geq d_n)$. If the order can be increased at all, A_r has to be nilpotent, which is a good sign since it is precisely the situation when spectral decoupling fails.

What is "better" ? "Better" means that

$$\dim(O') > \dim(O), \qquad O(O') = \text{ orbit of } A_r(A'_{r'})$$

By Jordan decomposition there are only finitely many nilpotent orbits. *This is the true reason why reduction stops after a finite number of steps.*

Canonical forms

$$B = D_1 z^{r_1} + D_2 z^{r_2} + \dots D_m z^{r_m} + z^{-1} C$$

where

(a) The D_j are nonzero diagonal matrices and commute with C.

(b) $r_1 < r_2 < \dots < r_m < -1$ are rational numbers.

The r_j are the *levels* of the system. r_1 is the *principal level*, also known as *Katz invariant.* They are invariants under all gauge transformations over the algebraic closure of $\mathbf{C} - z]$. The principal level is the largest value of the order for the connection matrices in the orbit of A under the gauge group $GL(n, \overline{\mathbf{C}_z})$. Let

$$G_z = GL(n, \mathbf{C}_z), \qquad \overline{G_z} = GL(n, \overline{\mathbf{C}_z})$$

Two connections are equivalent under $\overline{G_z}$ if and only if their canonical forms are equivalent under $GL(n, \mathbf{C})$. This is the ideal situation; the classifying problem with respect to the *infinite dimensional* gauge group is educed to the classifying problem over the *finite dimensional* $GL(n, \mathbf{C})$ acting on a finite dimensional variety consisting of the D_i, C.

Jacobson–Morozov

Nilpotents are harder to deal with than semisimple (=diagonalizable) transformations. One of the powerful techniques for this is the *Jacobson–Morozov* theorem which says (in the special case of $\mathfrak{gl}(n, \mathbf{C})$ that any nilpotent $Y \neq 0$ can be imbedded in an $\mathfrak{sl}(2)$ with basis H, X, Y such that

$$[H, X] = 2X, \, [H, Y] = -2Y, \, [X, Y] = H$$

Any two such $\mathfrak{sl}(2)$'s are conjugate under the centralizer of Y. Since the $\mathfrak{sl}(2)$–modules are semisimple, we have a semisimple object available for working with nilpotents. The price to pay is that we have to deal with the spectral theory of a noncommutative object, namely, $\mathfrak{sl}(2)$.

Let $\mathfrak{a} = \mathfrak{sl}(2)$. All $\mathfrak{a}$–modules (of finite dimension) are direct sums of irreducible modules. The irreducible ones are parameterized by integers $\lambda \geq 0$ ("highest weight"). The module with highest weight λ has a cyclic vector $v \neq 0$ such that $Hv = \lambda v, Xv = 0$, and the range of Y, which is the span of $Y^n v (n = 1, 2, \ldots, \lambda)$. In particular, in *any* module, the null space of X and the range of Y are complimentary linear subspaces. Moreover, as the null space of X is the span of the highest weight vectors in the module, it is clear that this null space is stable under H and H has only eigenvalues that are integers ≥ 0 on it.

Increasing the order of the connection

Alignment. We have

$$\mathfrak{sl}(n) = \mathfrak{z} \oplus [Y, \mathfrak{sl}(n)] \qquad (*)$$

where

$$\mathfrak{z} = \text{ centralizer of } X \text{ in } \mathfrak{sl}(n).$$

If

$$g = 1 + zT_1 + z^2T_2 + \dots$$

and

$$g[A] = A' = z^rY + z^{r+1}A'_{r+1} + \dots,$$

then, from $A'g = gA + \dot{g}$ we get

$$\begin{aligned} A'_{r+1} &= -[Y, T_1] + A_{r+1} \\ A'_{r+k} &= -[Y, T_k] + \sum_{j=1}^{k-1} T_j A_{r+k-j} + \dots \\ &\qquad - \sum_{j=1}^{k-1} A'_{r+k-j} T_j + (r+k+1) T_{r+k}. \end{aligned}$$

By $(*)$ we can choose T_1 such that $A'_{r+1} \in \mathfrak{z}$; and if $T_j (1 \leq j \leq k-1)$ have been chosen, we can choose T_k such that $A'_{r+k} \in \mathfrak{z}$. Thus we may assume that A itself is *aligned*, i.e.,

$$A_{r+m} \in \mathfrak{z} \qquad (m \geq 1).$$

Let $(Z_k)_{1\le k\le q}$ be a basis of $\mathfrak{z}$ such that

$$[H, Z_k] = 2\lambda_k Z_k \qquad (1 \le k \le q), A_{r+m} = \sum_k a_{r+m,k} Z_k.$$

for integers $2\lambda_k \ge 0$. Under conjugacy by

$$g_t = z^{-\frac{t}{2}H} \qquad (t > 0),$$

we have

$$z^r Y \to z^{r+t} Y, \qquad z^{r+m} Z_k \to z^{r+m-t\lambda_k} Z_k.$$

So the higher order terms move in the direction *opposite* to that of the leading term $z^r Y$. It is necessary to compare t with $m - t\lambda_k$ and so the quantities

$$\frac{m}{1+\lambda_k}$$

become significant in the choice of t so that $A' = g_t[A]$ has order $r + t$.

Let

$$\Lambda = \max(1+\lambda_k), \qquad M = \Lambda(|r|-1)$$
$$F = \{(m,k) \mid 1 \le m < M, 1 \le k \le q, a_{r+m,k} \neq 0\}$$
$$t = \inf_{(m,k)\in F} \frac{m}{1+\lambda_k} > 0 \ (F \text{ not empty }).$$

Lemma. *If F is empty, i.e., $A_{r+m} = 0$ for $1 \le m < M$, or if F is nonempty and $t \ge |r|-1$, then $A' = g_{|r|-1}[A]$ has a simple pole and so A is regular.*

Proof. For $m \ge M$,

$$r+m-(|r|-1)\lambda_k \ge r+\Lambda(|r|-1)-(\Lambda-1)(|r|-1) \ge -1.$$

If F is nonempty, and $1 \le m < M$, we have

$$\begin{aligned} r+m-(|r|-1)\lambda_k &\ge r+m-t\lambda_k \\ &\ge r+t(1+\lambda_k)-t\lambda_k \ge r+t \ge -1. \end{aligned}$$

Thus in either case A' has a simple pole and so A is regular.

By Lemma, if A is irregular, F is nonempty and $0 < t < |r| - 1$. Let $A' = g_t[A]$.

Lemma. *A' has order $r + t < -1$, and for its leading coefficient A'_{r+t} we have*

$$A'_{r+t} = Y + Z, \qquad 0 \neq Z \in \mathfrak{z}.$$

Proof. The same estimates as above are used. If $m \geq M$,

$$r + m - t\lambda_k > r + (|r| - 1)\Lambda - (|r| - 1)(\Lambda - 1) > r + t$$

while for $1 \leq m < M$,

$$r + m - t\lambda_k = r + t + [m - t(1 + \lambda_k)] \geq r + t.$$

In this estimate we have $> r + t$ if $m \neq t(1 + \lambda_k)$. By definition of t the set K of k for which there is m with $(m, k) \in F$ and $m = t(1 + \lambda_k)$ is nonempty; moreover for $k \in K$ there is only one m with $m = t(1 + \lambda_k$. If we write $c_k = a_{r+m,k}$, then

$$A'_{r+t} = Y + Z, \qquad Z = \sum_{k \in K} c_k Z_k \neq 0.$$

Increase of orbital dimension

If O_Z is the orbit of $Z \in \mathfrak{sl}(n)$ under the adjoint group and $d(Z) = \dim O_Z$, the basic result is the following. If $Y' = Y + Z$ where $0 \neq Z \in \mathfrak{z}$, then

$$d(Y') > d(Y).$$

Write $Y' = Y + \sum_k c_k Z_k$ and $h(t) = e^{tH}$. Then,

$$e^{2t} Y'^{h(t)} = Y + \sum_k c_k e^{(\lambda_k + 2)t} Z_k \to Y \qquad (t \to -\infty)$$

The left side is in the orbit of Y' for all t and so $Y \in \mathrm{Cl}\,(O_{Y'})$. Using the transversality of $\mathfrak{z}$ to the orbit of Y one can show that O_Y does not meet $Y + \mathfrak{z}$ close to Y except at Y; but the above calculation shows that if it meets $Y + \mathfrak{z}$ then it must meet it close to Y. Hence the orbit of Y meets $Y + \mathfrak{z}$ only at Y. So, the orbit of Y' is distinct from that of Y. This means that Y is strictly in the boundary of $O_{Y'}$, hence the dimensions obey the strict inequality above.

Problem B

Formal solutions are asymptotic to analytic solutions over sectors (Poincaré). If a formal canonical form B is fixed and the sector rotates, the analytic fundamental solution changes. This is the *Stokes phenomenon.* It is governed by the first cohomology of a certain sheaf St_B of unipotent groups. $H^1(St_B)$ classifies analytic systems with B as their formal canonical form. (Malgrange–Sibuya).

This is nonabelian cohomology and so it is not obvious that H^1 has additional structure.

Theorem (B–V). *$H^1(St_F)$ has the structure of affine space of dimension.*

Basic idea due to Deligne. One works with the sheaf of unipotent *group schemes* and so for any commutative **C**–algebra R we have a sheaf of groups $St_B(R)$ with first cohomology $H^1(St_B(R))$. The functor

$$R \longmapsto H^1(St_B(R))$$

is representable by affine space.

Problem C

It is a question of reduction of analytic *families* of systems with same formal structure. The viewpoint is the *relative* one. In previous studies this was not recognized and so the fact that the Jordan decomposition of a nilpotent changes discontinuously in families was an obstruction. In the relative view one starts with a domain R and studies $A \in \mathfrak{gl}(n, R)$ under gauge transformations $g \in GL(n, \overline{R_z})$ where

$$R_z = R[[z]][\frac{1}{z}]$$

and $\overline{R_z}$ is the algebraic closure of R_z. Let

$$A = A_r z^r + A_{r+1} z^{r+1} + \dots \ (r < -1. A_r \neq 0).$$

Nilpotents and their deformations over a general ring

To study the obstruction caused by the discontinuous behavior of the Jordan decomposition of nilpotents of a family, one takes the relative point of view and looks for Jordan decomposition of nilpotent matrices over the algebra R of germs of holomorphic functions at 0 or more general rings. We work in $\mathfrak{gl}(n, R)$ and ask for the classification of nilpotent elements under the action of $GL(n, R)$. There are *infinitely many orbits* and, for $n > 2$, even moduli. However one can construct *discrete invariants* which form a set with a natural *linear* order $<<$ satisfying a chain condition: *no sequence of strictly decreasing invariants is infinite.* Indeed, the discrete invariants are elements of $\mathbf{Z}^d_{\geq 0}$ with its natural *lexicographic* order.

The earlier reduction process still applies. $A'_{r'}$, if nilpotent, is now "better " in the following more refined sense: either it is nilpotent of orbital dimension $>$ that of A_r, or, it has the same orbital dimension but its discrte invariants are $<<$ those of A_r. The reduction process again terminates in a finite number of steps.

Remarks

For moduli questions we have to do reduction over multiparameter rings. This is technically more difficult. It can be proved that reduction is possible if R is Noetherian, integrally closed, and all projective modules over $R_z := R[[z]][\frac{1}{z}]$ are free. For A itself one must assume that the canonical form of A over the algebraic closure of R_z has the following form:

$$B = D_1 z^{r_1} + D_2 z^{r_2} + \ldots D_m z^{r_m} + z^{-1} C$$

where

(a) The D_j are diagonal but their entries are in R itself ("split"over R).

(b) If $(d_1, \ldots, d_m)$ and $(d'_1, \ldots, d'_m)$ are distinct elements in the spectrum of $(D_1, \ldots, D_m)$, then, for k the first integer i for which $d_i \neq d'_i$, then $d_k - d'_k$ is a unit of R.

References

The entire discussion above has been based on the work of Babbitt and Varadarajan. For a wide ranging account of this work as well as references that lend perspective to the whole subject, see my paper in *Bull. Amer. Math. Soc.*, **33** (1996) 1–42.

NOTES

Nilpotents over general rings

The family

$$N(t) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$$

is 0 for $t = 0$ but has Jordan form $N(1)$ for $t \neq 0$. In the *relative* theory N is viewed as a nilpotent of R^2 where $R = \mathbf{C}[[t]$ or $\mathbf{C}\{t\}$. What are the $GL(2, R)$–orbits of nilpotents? K is the quotient field of R.

$$U = \text{ kernel of } N \text{ in } R^2 = R \cdot e, \quad N[R^2] = t^m \cdot U$$

The integer $m \geq 0$ is an invariant of the $GL(2, R)$–orbit of N. So the Jordan forms are infinitely many and are

$$N_m(t) = \begin{pmatrix} 0 & 0 \\ t^m & 0 \end{pmatrix}.$$

In dimensions > 2 the orbits have "moduli".

Nilpotents in arbitrary dimensions

On K^n Jacobson–Morozov gives a canonical finite filtration $(W'_m) = (W'_{m,K})_{m\in\mathbf{Z}}$, $W'_m = 0$ for $|m| > M$, with $\overline{W'_m} = W'_m/W'_{m-1}$ such that

(1) $NW'_m \subset W'_{m-2}$

(2) $N^j : \overline{W'_m} \longrightarrow \overline{W'_{-m}}$ is an isomorphism

We put

$$W_m = W'_m \cap R^n$$

Then

$$M_j = \overline{W_{-j}}/N^j\overline{W_j}$$

is a finite torsion module for R. For R a *discrete valuation ring* with uniformizant t, M_j is of the form $\oplus_{1\leq i\leq q}(R/t^{a_i}R)$ for unique integers $a_1 \geq a_2 \leq \ldots \geq a_q \geq 0$. We put

$$d_j = d_j(N) = a_1 + \ldots$$

Let

$$\mathbf{d}(N) = (d_M(N), d_{M-1}(N), \ldots d_1(N)) \in \mathbf{Z}^M_{\geq 0}$$

These are the discrete invariants of N. We write $<<$ for the *lexicographic ordering* on $\mathbf{Z}^M_{\geq 0}$.

Semicontinuity of discrete invariants

Theorem (Upper semicontinuity of discrete invariants). *If $N(\tau)$ is a polynomial deformation of $N(0)$ ($\tau \in \mathbf{C}$), then, either orbital dimension of $N(0)$ is strictly less than that of $N(\tau)$ (for small nonzero τ), or else $N(\tau)$ for small nonzero τ is in the same orbit as $N(0)$ under $GL(n, K)$, and has constant $\mathbf{d}(N(\tau))$, but*

$$\mathbf{d}(N(\tau)) << \mathbf{d}(N(0)).$$

One can examine more closely what happens when there is equality above. For a class of deformations including the one occurring in redution theory over R one can show that equality cannot occur. Once this is done, the proof of reduction theory can be completed along earlier lines.

Stokes sheaf

Let A be meromorphic. The *Stokes sheaf* St_A is the sheaf on the unit circle S^1 such that for any arc Γ, $St_A(\Gamma)$ is the group of all matrices g of analytic functions on sectorial open sets based on Γ with the properties:

(a) $g \sim 1$ where ~ 1 means asymptotic to 1.

(b) $g[A] = A$.

Consider the set $\mathfrak{M}$ of all pairs (B, ξ) where B is meromorphic, ξ formal, with $\xi[B] = A$, modulo the equivalence relation $\approx$ defined by

$$(B, \xi) \approx (B', \xi')$$

if and only if there is meromorphic u such that

$$u[B] = B' \qquad \xi' \circ u = \xi$$

Theorem (Malgrange–Sibuya). *There is a canonical bijection of* $\mathfrak{M}$ *with* $H^1(St_A)$ *that takes* (A, id) *to the trivial class.*

Theorem. St_A *is a sheaf of complex algebraic unipotent groups on* S^1.

For determining the structure of the H^1 of the Stokes sheaf one needs to view St_A as a sheaf of unipotent group *schemes* over $\mathbf{C}$.