Liouvillian solutions of linear differential equations

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$$L(y) = \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1\frac{dy}{dx} + a_0\, y = 0$$

$a_i \in k$, differential field of characteristic 0
with algebraically closed field of constants C

K/k a Picard Vessiot extension

Differential Galois group $G \subseteq GL(n, C)$

Definition: K/k is Liouvillian if

$$k = K_0 \subset K_1 \subset \ldots \subset K_s = K$$

with $K_{i+1} = K_i(\nu_i)$, where either

- ν_i **algebraic over** K_i**,**
- $\nu_i' \in K_i$**,** $(\nu_i = \int u, u \in K_i)$
- **or** $\frac{\nu_i'}{\nu_i} \in K_i$ $(\nu_i = e^{\int u}, u \in K_i)$**.**

$$L(y) = L_{na}(L_{algebraic}(y)) = L_{nl}(L_{liouvillian}(y)$$

$\Rightarrow$ suppose $L(y)$ is irreducible.

Kolchin

- $L(y) = 0$ **has algebraic solutions** $\Leftrightarrow G$ **is finite**
- $L(y) = 0$ **has Liouvillian solutions** $\Leftrightarrow G^\circ$ **is solvable**

$$
\begin{array}{ccc}
K & \longleftrightarrow & \{\mathrm{id}\} \\
\cup & & \cap \\
\frac{z'}{z} \in F & \longleftrightarrow & G^\circ = \left\{ \begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,n} \\ 0 & c_{2,2} & \dots & c_{2,n} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & c_{n,n} \end{pmatrix} \right\} \\
\cup & & \cap \ < \infty \\
k & \longleftrightarrow & G
\end{array}
$$

$\Rightarrow \exists\, z = e^{\int u}$ where $[k(u) : k] = m < \infty$.

Algorithm: find $u = \frac{z'}{z}$

1. Bound m.
2. Compute the minimal polynomial of u.

Kovacic for $n = 2$, $m \in \{1, 2, 4, 6, 12\}$

Singer: $m \leq \left(\sqrt{8n} + 1\right)^{2n^2} - \left(\sqrt{8n} - 1\right)^{2n^2}$

$Q \in k[U]$ minimal polynomial of $u = \dfrac{z'}{z}$

$$Q \;=\; \left(U - \frac{z_1'}{z_1}\right)\left(U - \frac{z_2'}{z_2}\right)\ldots\left(U - \frac{z_m'}{z_m}\right).$$

$$Q \;=\; U^m - \frac{(z_1 z_2 \cdots z_m)'}{z_1 z_2 \ldots z_m} U^{m-1} + \ldots + (-1)^m \frac{z_i' z_2' \cdots z_m'}{z_1 z_2 \ldots z_m}$$

$\Rightarrow \quad \exists$ solutions $z_1, \ldots, z_m$ such that

$$\frac{(z_1 z_2 \cdots z_m)'}{z_1 z_2 \ldots z_m} \in k$$

$$\begin{aligned} P \quad = \quad & (z_1 z_2 \ldots z_m)\, U^m \\ & - (z_1' z_2 \cdots z_m + \ldots + z_1 \cdots z_{m-1} z_m')\, U^{m-1} \\ & + \ldots + (-1)^m\, (z_1' z_2' \cdots z_m') \end{aligned}$$

Revisiting Kovacic's algorithm $L(y) = y'' + a_0 y = 0$ and $m = 2$

$$(Z_1 Z_2)\, U^2 - (Z_1' Z_2 + Z_1 Z_2')\, U + (Z_1' Z_2')$$

$$Z_i'' = -a_0 Z_i$$

$$\begin{array}{lllll} S & = & Z_1 Z_2 & & \\ S' & = & & Z_1' Z_2 + Z_1 Z_2' & \\ S'' & = & -2a_0\, Z_1 Z_2 & & +2\, Z_1' Z_2' \\ S''' & = & -2a_0'\, Z_1 Z_2 & -4a_0 (Z_1' Z_2 + Z_1 Z_2') & \end{array}$$

$$\begin{pmatrix} S \\ S' \\ S'' \\ S''' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2a_0 & 0 & 2 \\ -2a_0' & -4a_0 & 0 \end{pmatrix} \begin{pmatrix} Z_1 Z_2 \\ Z_1' Z_2 + Z_1 Z_2' \\ Z_1' Z_2' \end{pmatrix}$$

$$\Rightarrow \qquad S''' + 4a_0 S' + 2a_0' S = 0$$

S is solution of $L^{ⓢ 2}(y) = y''' + 4a_0 y' + 2a_0' y$

$$\begin{pmatrix} S \\ S' \\ S'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2a_0 & 0 & 2 \end{pmatrix} \begin{pmatrix} Z_1 Z_2 \\ Z_1' Z_2 + Z_1 Z_2' \\ Z_1' Z_2' \end{pmatrix}$$

$$P \;=\; S\,U^2 \;-\; S'\,U + \left(a_0 S + \frac{S''}{2} \right)$$

If $S \in k$ (rational solution) then $P \in k[U]$
(Always possible for $m \in \{1, 4, 6, 8, 12\}$)

Theorem: if $\frac{S'}{S} \in k$, then

$$U^2 \;-\; \frac{S'}{S}\,U + \frac{\left(a_0 s + \frac{S''}{2} \right)}{S} \;\in\; k[U]$$

Let $y_1, \ldots, y_n$ basis of solutions with

$$\forall \sigma \in G, \quad \sigma(y_j) = \sum_{i=1}^{n} \alpha_{i,j} y_i, \qquad \alpha_{i,j} \in k$$

G acts on $C[Y_1, \ldots, Y_n]$ via $\sigma(Y_j) = \sum_{i=1}^{n} \alpha_{i,j} Y_i$

$S \in C[Y_1, \ldots, Y_n]$ is semi-invariant iff
$\forall \sigma \in G, \quad \sigma \cdot S = \psi_I(\sigma) \cdot S$, $\psi_I(\sigma) \in \mathcal{C}$

$$\begin{aligned} \Phi: \mathcal{C}[Y_1, \ldots, Y_n] &\rightarrow K \\ Y_i &\mapsto y_i \end{aligned}$$

1. Linear forms are sent to solutions of $L(y) = 0$.
2. S Semi-invariants $\mapsto f$ where $f'/f \in k$

$L(y) = 0$ has a Liouvillian solution $\Leftrightarrow$ $G \subseteq GL(n, C)$ has a semi-invariant which factors into linear forms

Let $\sigma \in G$, $S = Z_1 Z_2 \cdots Z_m \in k[Y_1, \ldots, Y_n]$.

$$\sigma(S) = \sigma(Z_1) \ldots \sigma(Z_m) = c_\sigma Z_1 \cdots Z_m$$

$$\Rightarrow \quad \sigma(Z_i) = c_{i,j} Z_j \quad \Rightarrow \quad \sigma\left(\frac{Z_i'}{Z_i}\right) = \frac{Z_j'}{Z_j}$$

NB: $n = 2$ any semi-invariant will do !

Algorithm:

1. find $S \in C[Y_1, \cdots, Y_n]$ homogeneous of degree m such that
   (a) $\Phi(P) = f$ with $f'/f \in k$
   (b) P factors into linear forms
2. Construct minimal polynomial whose roots are the values of the factors.

Problems/Limitations

1. Computation of high degree semi-invariants
2. Has to locate semi-invariant that factor into linear forms in a space of semi-invariant of high dimension
3. Matrix A may not be invertible

$n = 3$: $m \in \{1, 3, 6, 9, 21, 36\}$. Semi-invariant of degree 36 that factor is in space of dimension 5 ($\Rightarrow$ absolute factorisation with 5 parameters).

O. Cormier: $n = 6$ and G the Hall-Janko group J_2, $m = 3780$ with 441 parameters

$G \subseteq GL(n, C)$ irreducible is imprimitive if
$\exists \oplus_{i=1}^{s>1} V_i = V$ where $\forall \sigma \in G$, $\sigma(V_i) = V_j$.

1. G is primitive
   (a) $|G/Z(G)| = \infty$, no liouvillian solution
   (b) $|G/Z(G)| < \infty$ $\exists$ liouvillian solution
2. G is imprimitive
   (a) $s = n$ then G is monomial and $m = n$
   (b) $s < n$, $G_i = stab_G(V_i)$ is primitive on V_i
   $|G_i/Z(G_i)| < \infty \Leftrightarrow \exists$ liouvillian solution

$n = 2$ then $m \in \{2\} \cup \{4, 6, 12\}$

$n = 3$ then $m \in \{3\} \cup \{6, 9, 21, 36\}$

$n = 4$ then $m \in \{4\} \cup \{8, 12, 24, 20\} \cup$
. $\{5, 8, 10, 12, 16, 20, 24, 40, 48, 60, 72, 120\}$

$n = 5$ then $m \in \{5\} \cup \{6, 10, 15, 30, 40, 55\}$

Suppose $G \subseteq SL(n,C)$, e.g. $a_{n-1} = 0$.

$$\tilde{y} = y \cdot \exp\left(-\frac{\int a_{n-1}}{n}\right)$$

1. primitive:
   (a) $|G| = \infty$, no liouvillian solution
   (b) $|G| < \infty$, then G is in a finite list
   and all solution are algebraic
2. imprimitive:
   (a) $s = n$ then G is monomial and $m = n$
   (b) $s < n$, $G_i = stab_G(V_i)$ is primitive on V_i
   $|G_i/Z(G_i)| < \infty \Leftrightarrow \exists$ liouvillian solution

primitive groups:

$n = 2$, $G \in \{SL(2,C)\}$ or $\{A_4^{SL_2}, S_4^{SL_2}, A_5^{SL_2}\}$

$n = 3$, $G \in \{PSL(2,C), SL(3,C)\}$ or

$\{F_{36}^{SL_2}, H_{72}^{SL_2}, H_{216}^{SL_2}, A_5, A_5 \times C_3, G_{168}, G_{168} \times C_3, A_6\}$

Third Order Algorithm

1. **Reducible case:** factor $L(y)$
2. **If $L^{\circledS 3}(y)$ is of order** 7: $2a_0 = a_1'$.
   $L = \tilde{L}^{\circledS 2}$ where $\tilde{L}(y) = y'' + \frac{a_1(x)}{4} y$
3. **Imprimitive case:** If $L^{\circledS 3}(y) = 0$ has
   (a) two rational solutions $\Rightarrow G \cong G_{27}$.
   (b) two exponential solutions $G \cong G_{54}$
   (c) One rational and ≥ 2 exponential solutions $\Rightarrow G \cong G_{81}$
   (d) one rational solution and $L^{\circledS 3}(y)$ of order 9 or a semi-invariant $\Rightarrow G \cong G_{162}$.
   (e) **Generic case:** One possible S and formula holds.
4. **Primitive case:** try the 8 finite groups.

We end up either with a solution, a second order equation or a list of 12 possible finite groups

Example

$$\frac{d^3}{dx^3}y(x) + \frac{123x^2 - 103x + 108}{144x^2\left(x^2 - 2\,x + 1\right)}\frac{d}{dx}y(x)$$

$$- \frac{748x^3 - 937x^2 - 648 + 1605x}{864x^3\left(x^3 - 3\,x^2 + 3x - 1\right)}\,y(x)$$

$L^{\circledS 3}(y) = 0$ of order 10 one exponential solution $s = x^{7/2} - 2x^{5/2} + x^{3/2}$

Solving the 10×10 system for $s = x^{7/2} - 2x^{5/2} + x^{3/2}$ we get

$$U^3 - \frac{7\,x - 3}{2x\,(x-1)}\,U^2 + \frac{187\,x^2 - 159\,x + 36}{48\,x^2\,(x-1)^2}\,U$$

$$- \frac{1156\,x^3 - 1467\,x^2 + 675\,x - 108}{864\,x^3\,(x-1)^3}$$

Known finite groups

z with $[k(z'/z):k] = m \Rightarrow \exists d_j \in k$ and $i \in \mathbb{N}$

$$\mathbf{Y}^{i\cdot m} + d_{m-1}\mathbf{Y}^{i\cdot(m-1)} + \ldots + d_1\mathbf{Y}^i + d_0$$

Note: Multiples of z have same (m, i)

For $\sigma, \tau \in G$ we note that:

1. $\sigma \in \mathrm{Stab}_G(z'/z) \Leftrightarrow \sigma(z) = c_\sigma \cdot z$.
2. $\left(\tau\sigma\tau^{-1}\right)(\tau(z)) = c_\sigma \cdot \tau(z)$
3. $\tau(z)$ and z have the same (m, i) since

$$\sum_{j=1}^{m} b_i \left(\frac{z'}{z}\right)^j = 0 \;\Rightarrow \sum_{j=1}^{m} b_i \left(\frac{\tau(z)'}{\tau(z)}\right)^j = 0$$

$$\sum_{j=1}^{m} d_i z^{j\cdot i} = 0 \;\Rightarrow \sum_{j=1}^{m} d_i \left(\tau(z)\right)^{j\cdot i} = 0$$

4. Eigenvectors of σ and $\tau\sigma\tau^{-1}$ for the same eigenvalue have the same (m, i) if eigenspace of dim 1.

$$\frac{d^3y}{dx^3} + \frac{7x-3}{x(x-1)}\frac{d^2y}{dx^2} + \frac{2(138x^2-115x+7)}{27x^2(x-1)^2}\frac{dy}{dx}$$

$$+\frac{10(162x^2-28x-7)}{729x^3(x-1)^2}y.$$

	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{8}{9}$
ram	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{7}{9}$	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{8}{9}$
type	1	1	$\frac{4}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{2}{9}$
i	3	9						
m	9	3						
i	3	9						
m	9	3						
i	3	9	9	9	9	9	9	9
m	9	3	3	3	3	3	3	3

at 0 the formal solutions are:

$$s_1 = x^{\frac{-5}{9}} \cdot (1 + \frac{5}{9}x + \ldots),\ s_2 = x^{\frac{-2}{9}} \cdot (-1 - \frac{11}{9}x + \ldots)$$

$$s_3 = x^{\frac{-2}{9}} \cdot (x + \ldots)$$

$(s_1)^9$ is solution of

$$P = \sum_{i=0}^{3} b_j Y^j$$

$$b_j = \frac{\left(\sum_{s=0}^{2(3-j)} \gamma_{j,s} x^s\right)}{x^{5(3-j)}(x-1)^{6(3-j)}}.$$

$\Rightarrow$ linear system for $\gamma_{j,s}$

$$\left(Y^9\right)^3 - \frac{x^2 - 9x + 9}{3^2 x^5 (x-1)^6} \left(Y^9\right)^2$$

$$+ \frac{1}{3^5 x^6 (x-1)^{12}} Y^9 - \frac{1}{3^9 x^9 (x-1)^{18}}$$

Constructing a linear differential equation for a given finite group

Given $G \in GL(n, \mathbb{C})$ with $|G| < \infty$, find

$$L(y) = \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 \, y = 0$$

Riemann-Hilbert Problem, *Hilbert's 16th Problem*

Solution in (C. Tretkoff and M. Tretkoff, Am. J. Math., 101, 1979), but no construction

Different methods infinite groups: Kovacic, Singer, Mitschi, ...

From polynomials to differential equations

$$P \;=\; \mathbf{Y}^{54} + b_1\,\mathbf{Y}^{36} + b_2\,\mathbf{Y}^{18} + b_3$$

$$C(x)(z) \;=\; C(x)[Y]/(P), \qquad [C(x)(z) : C(x)] = 54$$

$$54z^{53}z' + b_1'\,z^{36} + 36b_1\,z^{35}z' + b_2'\,z^{18} + 18b_2\,z^{17}z' + b_3' \;=\; 0$$

$$z' = -\frac{b_1'\,z^{36} + b_2'\,z^{18} + b_3'}{54\,z^{53} + 36\,b_1\,z^{35} + 18\,b_2\,z^{17}} \in C(x)(z)$$

$$\Rightarrow \quad z, z', z'', \ldots, z^{(54)} \in C(x)(z)$$

$$\frac{d^3y}{dx^3} - \frac{8x-3}{x\,(x-1)}\,\frac{d^2y}{dx^2} + \frac{3264\,x^2 - 2452\,x + 729}{108\,x^2\,(x-1)^2}\,\frac{dy}{dx}$$
$$-\frac{71280\,x^3 - 80410\,x^2 + 47466\,x - 10935}{1458\,x^3\,(x-1)^3}\,y$$

G-invariant space

If $\{y_1, \ldots, y_n\}$ spann a G-invariant $\mathbb{C}$-linear space, then

$$L(y) = \frac{|Wr(y_1, y_2, \ldots, y_n, y)|}{|Wr(y_1, y_2, \ldots, y_n)|}$$

$$L(y) \;=\; \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1\frac{dy}{dx} + a_0\, y \;=\; 0$$

$$a_i = (-1)^i \frac{|Wr_i(y_1, y_2, \ldots, y_n)|}{|Wr(y_1, y_2, \ldots, y_n)|}$$

NB: The regular representation contains all irreducible representations...

The Hurwitz example

Klein quadric $x^3y + y^3z + z^3x$ of genus 3 with automorphism group G_{168}.

Math. Ann. 14, (1878/79). Footnote page 21:

"Sie muss sich auch durch eine lineare Differentialgleichung dritter Ordnung lösen lassen; wie hat man dieselbe aufzustellen?"

A. Hurwitz, Math. Ann. 26, (1886)

$\mathcal{C} \to \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ ramified in $0, 1, \infty$

$\omega_1 = f_1 dt, \omega_2 = f_2 dt, \omega_3 = f_3 dt$

$\Rightarrow \quad \{f_1, f_2, f_3\}$ spann a G_{168} invariant space

$$\begin{aligned} H(y) \quad = \quad & \frac{d^3y}{dx^3} - \frac{7x-4}{x\,(x-1)}\,\frac{d^2y}{dx^2} \\ & + \frac{\frac{72}{7}(x^2-x) - \frac{20}{9}(x-1) + \frac{3}{4}x}{x^2\,(x-1)^2}\,\frac{dy}{dx} \\ & - \frac{\frac{792}{7^3}(x-1) + \frac{5}{8} + 263}{x^2\,(x-1)^2}\,y \end{aligned}$$

Galois covering

Given: a finite group $G \subset GL(n, \mathbb{C})$ generated by $\sigma_1, \ldots, \sigma_s$ with $(\sigma_1 \cdots \sigma_{s-1})^{-1} = \sigma_s$.

There exists $a_1, a_2, \ldots, a_s$ on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and a Galois covering $\mathcal{C} \to \mathbb{P}^1 \setminus \{a_1, \ldots, a_s\}$ with group G, unramified outside $a_1, a_2, \ldots, a_s$ and the " Deck" action at a_i given by σ_i

Analytic construction of $\mathcal{C} \to \mathbb{P}^1$ does not lead to a determination of the algebraic curve $\mathcal{C}$.

But we can compute information on $\mathcal{C} \to \mathbb{P}^1$ or the corresponding Galois extension $K/\mathbb{C}(x)$

Idea: find a G-invariant subspace in $K/\mathbb{C}(x)$ corresponding to the correct character of G in the linear space of holomorphic one forms.

Characters and holomorphic one forms

Let e_i be the order of σ_i.

$$g(\mathcal{C}) = 1 + \frac{|G|}{2}\left(1 - \sum_{j=1}^{s} \frac{1}{e_j}\right)$$

$$\mathcal{D} = \mathbf{1} - \chi_{reg} + \sum_{j=1}^{s} \sum_{k=1}^{e_j - 1} \frac{k}{e_j} Ind_{<\sigma_j>}^{G} e^{\frac{2i\pi}{k}}$$

character χ appears in homomorphic one forms $\Leftrightarrow < \mathcal{D}, \chi > \geq 1$

1. G finite $\Rightarrow L(y)$ Fuchsian and
   . $Gal(G) =$ monodromy group

$$\begin{pmatrix} e^{2i\pi\alpha_1} & 0 & \cdots & 0 & 0 \\ 0 & e^{2i\pi\alpha_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & e^{2i\pi\alpha_n} \end{pmatrix}$$

$$s_i = t^{\alpha_i} \cdot (\beta_0 + \beta_1 t + \ldots)$$

   - $\alpha_i \in \mathbb{Q}$ and fixed up to integer by the generators of $G \subset GL(n, \mathbb{C})$
   - at one singularity $\alpha_i \in \mathbb{Q}$ distinct

2. sdt is holomorphic one form $\Rightarrow$
   - $-1 < \alpha$ at finite points
   - $1 < \alpha$ at ∞

3. Fuchs Relation for s generators

$$\sum exponents = \frac{(s-2)n(n-1)}{2}$$

$\Rightarrow$ Finite number of possible exponents

If $L(y)$ has ν singular points (including ∞), then $L(y)$ is determined by its exponents up to

$$\frac{1}{2}(n-1)(n \cdot s - 2n - 2)$$

accessory parameters.

imprimitive group G_{54}

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -\omega^5 - \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & -\omega^5 - \omega^2 \end{pmatrix}$$

$$\omega^6 + \omega^3 + 1 = 0$$

```
Class |   1 2    3    4 5 6 7 8   9  10
Size  |   1 9    1    1 6 6 6 6   9   9
Order |   1 2    3    3 3 3 3 3   6   6
---------------------------------------
p  =  2   1 1    4    3 5 6 7 8   4   3
p  =  3   1 2    1    1 1 1 1 1   2   2
---------------------------------------
X.1   +   1 1    1    1 1 1 1 1   1   1
X.2   +   1-1    1    1 1 1 1 1  -1  -1
X.3   +   2 0    2    2 2-1-1-1   0   0
X.4   +   2 0    2    2-1 2-1-1   0   0
X.5   +   2 0    2    2-1-1-1 2   0   0
X.6   +   2 0    2    2-1-1 2-1   0   0
X.7   0   3-1  3*J-3-3*J 0 0 0 0  -J 1+J
X.8   0   3 1  3*J-3-3*J 0 0 0 0   J-1-J
X.9   0   3-1-3-3*J  3*J 0 0 0 0 1+J  -J
X.10  0   3 1-3-3*J  3*J 0 0 0 0-1-J   J
```

J = RootOfUnity(3), Look for character 7

$$G = < g_1, g_2, g_3, g_4 >, \qquad g_1 g_2 g_3 g_4 = id$$

$$g_i \in C_{10} \quad \Rightarrow \quad \{\frac{5}{6}, \frac{5}{6}, \frac{1}{3}\}$$

minimal exponents are:

$$\{\frac{-1}{6},\frac{5}{6},\frac{-2}{3}\},\{\frac{-1}{6},\frac{5}{6},\frac{-2}{3}\},\{\frac{-1}{6},\frac{5}{6},\frac{-2}{3}\},\{\frac{11}{6},\frac{17}{6},\frac{4}{3}\}$$

$$\sum exponents = \frac{(s-2)n(n-1)}{2} = 6$$

$\Rightarrow$ no apparent singularities and exponents given above

$$\frac{d^3}{dx^3}y(x) + \left(\frac{p_{10}}{x} + \frac{p_{11}}{x-1} + \frac{p_{12}}{x+1}\right)\frac{d^2}{dx^2}y(x) +$$
$$\left(\frac{p_{20}}{x^2} + \frac{p_{21}}{(x-1)^2} + \frac{p_{22}}{(x+1)^2} + \frac{q_{21}\,x + q_{20}}{x\,(x-1)\,(x+1)}\right)\frac{d}{dx}y(x) +$$
$$\left(\frac{p_{30}}{x^3} + \frac{p_{31}}{(x-1)^3} + \frac{p_{32}}{(x+1)^3} + \frac{q_{33}\,x^3 + q_{32}\,x^2 + q_{31}\,x + q_{30}}{x^2\,(x-1)^2\,(x+1)^2}\right)y(x)$$

$$x^3 + (-3+p_{10})\,x^2 + (2-p_{10}+p_{20})\,x + p_{30} = 0$$
$$x^3 + (p_{11}-3)\,x^2 + (p_{21}-p_{11}+2)\,x + p_{31} = 0$$
$$x^3 + (p_{12}-3)\,x^2 + (p_{22}-p_{12}+2)\,x + p_{32} = 0$$
$$x^3 + (3-p_{12}-p_{10}-p_{11})\,x^2$$
$$+(2-p_{10}-p_{11}-p_{12}+p_{20}+p_{21}+p_{22}+q_{21})\,x$$
$$-p_{32}-p_{31}-p_{33}-p_{30} = 0$$

$$p_{20} = \frac{5}{12}, p_{32} = -\frac{5}{54}, p_{12} = 3, p_{22} = \frac{5}{12},$$

$$q_{21} = \frac{103}{6}, q_{33} = \frac{389}{54}, p_{31} = -\frac{5}{54},$$

$$p_{11} = 3, p_{21} = \frac{5}{12}, p_{30} = -\frac{5}{54}, p_{10} = 3$$

formal solutions at $x = 0$

$$s_1 = \frac{-\left(-120\, q_{30} + 70\, b^2 + 20\, q_{20}\, b + 70\, b\right) \ln(x)}{180\, b^2\, x^{\frac{1}{6}}} - \frac{(-36\, q_{30} - 15\, b^2 - 30\, q_{20}\, b - 15\, b)x + O\left(x^2\right)}{180\, b^2\, x^{\frac{1}{6}}}$$

$$s_2 = \frac{\left(x + O\left(x^2\right)\right)}{x^{\frac{1}{6}}}$$

$$s_3 = \frac{1 + \left(4\, \frac{q_{30}}{b^2} - \frac{40}{3} - \frac{40}{3}\, b^{-1} - 8/3\, \frac{q_{20}}{b}\right) x + O\left(x^2\right)}{x^{-2/3}}$$

$$q_{30} = \frac{7}{12}\, b^2 + 1/6\, q_{20}\, b + \frac{7}{12}\, b$$

Log's give all coefficients but b. Invariant of degree 3 gives b:

$$Dx^3 + 3\,\frac{\left(3\,x^2 - 1\right)}{x\,\left(x-1\right)\left(x+1\right)}\,Dx^2$$

$$+\frac{\left(221\,x^4 - 206\,x^2 + 5\right)}{12\,x^2\left(x-1\right)^2\left(x+1\right)^2}\,Dx$$

$$+\frac{\frac{187}{27}\,x^6 - \frac{673}{54}\,x^4 + \frac{127}{27}\,x^2 + \frac{5}{54}}{x^3\left(x-1\right)^3\left(x+1\right)^3}$$