

# Kolchin Seminar

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## Linear Differential Ideals

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# Notations

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- $\mathcal{F}$ , a differential field, characteristic zero
- $\Delta = \{ \delta_1, \dots, \delta_m \}$  commuting derivations
- $\mathcal{C}$ , constant subfield of  $\mathcal{F}$
- $Y : y_1, \dots, y_n$  differential indeterminates
- $\Theta = \{ \theta = \delta_1^{e_1} \dots \delta_m^{e_m} \}$ ,  $\Theta Y = \{ \theta y_j \}$
- $\mathcal{R} = \mathcal{F}\{ y_1, \dots, y_n \} = \mathcal{F}[\Theta Y]$  differential polynomial ring
- System of PADE (partial algebraic differential equations)

$$F_i(y_1, \dots, y_n) = 0, \quad i = 1, \dots, k$$

- $\Phi = \text{set of } F_1, \dots, F_k$
- $\mathfrak{a} = [\Phi]$

# Differential Modules

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- $\mathcal{A}$  differential ring
- $\mathcal{V} = (V, D)$ ,  $V$  is  $\mathcal{A}$ -module,  $D$  action of  $\Delta$  on  $V$ 
  1.  $D(\delta, a\eta) = (\delta a)\eta + aD(\delta, \eta)$ ,
  2.  $D(\delta, \eta + \xi) = D(\delta, \eta) + D(\delta, \xi)$ ,
  3.  $D(\delta, D(\delta', \eta)) = D(\delta', D(\delta, \eta))$ .
- $\delta v$  for  $D(\delta, v)$
- If  $\mathcal{A}$  is a differential field, ...

# Linear Differential Polynomials

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- $F \in \mathcal{R}$  is *linear* if

$$F(y_1, \dots, y_n) = a_0 + \sum_{i=1}^q a_i \theta_i y_{k_i} \quad (1)$$

and *linear homogeneous* if  $a_0 = 0$ .

- $\mathcal{R}_1 = \mathcal{F}\{y_1, \dots, y_n\}_1$ , differential vector space
- $\mathfrak{p}$  is *linear* if  $\mathfrak{p} = [\Lambda]$ ,  $\Lambda \subset \mathcal{R}_1$ .
- $\mathfrak{p}$  linear  $\Rightarrow$   $\mathfrak{p}$  prime and homogeneous
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$$\mathcal{L} = \mathfrak{p} \cap \mathcal{R}_1 = \sum_{\theta \in \Theta, L \in \Lambda} \mathcal{F} \cdot \theta L \quad (2)$$

# Linear Differential Ideals

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**Proposition** *The mapping  $\mathfrak{p} \mapsto \mathfrak{p} \cap \mathfrak{R}_1$ ,  $\mathfrak{p}$  linear  $\Delta$ -ideal of  $\mathfrak{R}$ , is bijective; the inverse  $\mathcal{L} \mapsto [\mathcal{L}] = (\mathcal{L})$ ,  $\mathcal{L}$   $\Delta$ -subspace of  $\mathfrak{R}_1$ .*

- Ranking:  $u \leq \theta u$ , and  $u \leq v \Rightarrow \theta u \leq \theta v$  for any  $u, v \in \Theta Y, \theta \in \Theta$ .  
Every ranking is a well-ordering.
- $F \in \mathfrak{R}$ ,  $u_F$  is leader (highest ranked derivative)
- $u_{\mathcal{L}} =$  set of all  $\theta y_j$  such that it is a leader of some  $F \in \mathcal{L}$
- If  $u \in u_{\mathcal{L}}$ , there exists a *unique*  $L_u \in \mathcal{L}$  of the form

$$u + \sum a_{uv}v \quad (a_{uv} \in \mathfrak{F}) \tag{3}$$

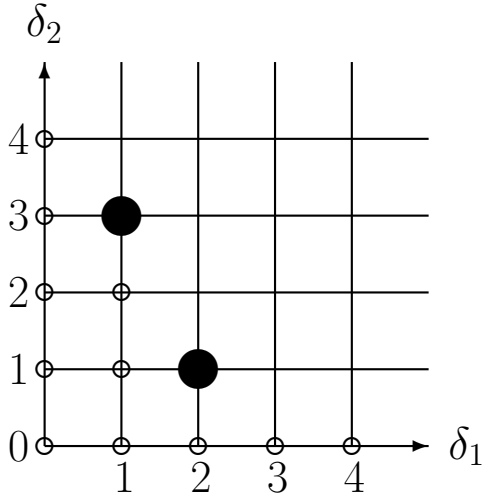
where  $v \notin u_{\mathcal{L}}$  and  $v$  lower rank than  $u$ .

**Proposition** *Fix orderly ranking. Every linear  $\mathfrak{p}$  has a unique canonical characteristic set with elements of the form (3).*

# Linear Dimension

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**Proposition** *The complement of  $u_{\mathcal{L}}$  in  $\Theta Y$  is  $\mathcal{F}$ -basis of  $\mathcal{R}_1/\mathcal{L}$ . Furthermore, if the ranking is orderly, and if  $\mathbf{B} : B_1, \dots, B_k$  is the canonical characteristic set of  $\mathfrak{p}$ , where the leader of  $B_i$  is  $u_i$ , then the complement of  $u_{\mathcal{L}}$  is the set of derivatives  $v \in \Theta Y$  that is not a derivative of any  $u_i, i = 1, \dots, k$ .*



Leaders  $\delta_1^2 \delta_2 y, \delta_1 \delta_2^3 y$

- The dimension of  $\mathcal{R}_1/\mathcal{L}$  is the *linear dimension* of  $\mathfrak{p}$ .

# Finite Dimensional $\Delta$ vector space

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- $\mathcal{V} = (V, D)$  differential vector spaces
- $\alpha : \alpha_1, \dots, \alpha_d$  basis of  $V$
- $A = (A_1, \dots, A_m) \in gl(d, \mathcal{F})^m$
- $\delta_i \alpha = A_i \alpha$ , in details:

$$\delta_i \alpha_k = \sum_{k'=1}^d a_{ikk'} \alpha_{k'} \quad (4)$$

where  $a_{ikk'} \in \mathcal{F}$ .

- $A_i = (a_{ikk'})_{1 \leq k \leq d, 1 \leq k' \leq d}$
- $A_i$  is matrix representation of  $\delta_i$  with respect to basis  $\alpha$

# Integrability Conditions

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- $\delta_i \delta_j = \delta_j \delta_i \Rightarrow$

$$\delta_i A_j + A_j A_i = \delta_j A_i + A_i A_j, \quad (1 \leq i, j \leq m). \quad (5)$$

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$$\delta_i A_j - \delta_j A_i = [A_i, A_j], \quad (1 \leq i, j \leq m).$$

- The set of equations in (5) is called the *integrability conditions*.
- The set of all  $m$ -tuple of matrices satisfying the integrability conditions is denoted by  $\mathbb{I}$  or  $\mathbb{I}(d, \mathfrak{F})$



# Characterization of Structure Matrices

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- $A = (A_1, \dots, A_m)$   $d \times d$  matrices in  $gl(n, \mathcal{F})$
- $A_i = (a_{ikk'})$
- $\mathcal{L} : L_{ik} = \delta_i y_k - \sum_{k'=1}^d a_{ikk'} y_{k'}$
- $\mathfrak{p} = [\mathcal{L}]$

**Theorem**  $\mathfrak{p}$  has linear dimension  $\leq d$ . Equality holds if and only if  $A \in \mathbb{I}$ . If this is the case,  $\mathfrak{p} \cap \mathcal{F}[y_1, \dots, y_n] = (0)$ .

# Loewy Action

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- $GL(d, \mathcal{F})$  acts on  $gl(d, \mathcal{F})^m$
- $G \in GL(d, \mathcal{F})$  and  $A = (A_1, \dots, A_m) \in gl(d, \mathcal{F})^m$
- Then  $G(A) = (B_1, \dots, B_m)$  where

$$B_i = \delta_i G \cdot G^{-1} + GA_i G^{-1}, \quad (1 \leq i \leq m). \quad (6)$$

- If  $A, B \in gl(d, \mathcal{F})^m$ , we say that  $B$  is *Loewy similar to*, or is a *Loewy conjugate of*  $A$  if there exists a  $G \in GL(d, \mathcal{F})$  such that  $G(A) = B$ .
- Loewy similarity is an equivalence relation on  $gl(d, \mathcal{F})^m$  and indeed, on  $\mathbb{I}$ .

**Proposition** *If  $A \in \mathbb{I}$ , and  $B = G(A)$ , then  $B \in \mathbb{I}$ .*

# Classification

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**Theorem** *Let  $\mathcal{D}^d(\mathcal{F})$  be the set of isomorphism classes of differential vector spaces with dimension  $d$  over  $\mathcal{F}$ . Let  $\mathcal{E}^d(\mathcal{F})$  be the set of equivalent classes of elements of  $\mathbb{I}$  under Loewy similarity. Then there is a bijection between  $\mathcal{D}^d(\mathcal{F})$  and  $\mathcal{E}^d(\mathcal{F})$ .*

- Let  $\mathcal{V} = (V, D)$  be a representative of an element  $\nu \in \mathcal{D}^d(\mathcal{F})$ .
- Fix a basis  $\alpha : \alpha_1, \dots, \alpha_d$  of  $V$  over  $\mathcal{F}$
- $A = (A_1, \dots, A_m)$  be the representation of  $D$  with respect to  $\alpha$ . That is,  $\delta_i \alpha = A_i \alpha$ .
- $\sphericalangle A \sphericalangle$  the equivalent class of  $A$  under Loewy similarity.
- Define  $\psi : \mathcal{D}^d(\mathcal{F}) \longrightarrow \mathcal{E}^d(\mathcal{F})$  by  $\psi(\nu) = \sphericalangle A \sphericalangle$
- $\psi$  is the bijection

# Homomorphisms

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- $\mathcal{V} = (V, D)$ ,  $\mathcal{V}' = (V', D')$  differential vector spaces
- $\sigma : V \longrightarrow V'$  linear map
- $\alpha : \alpha_1, \dots, \alpha_d$  basis of  $V$
- $\beta : \beta_1, \dots, \beta_{d'}$  basis of  $V'$
- $A = (A_1, \dots, A_m) \in gl(d, \mathfrak{F})^m$ ,  $\delta_i \alpha = A_i \alpha$
- $B = (B_1, \dots, B_m) \in gl(d', \mathfrak{F})^m$ ,  $\delta_i \beta = B_j \beta$
- $A, B$  are matrix representations of  $D, D'$  respectively.
- *homomorphism* if  $\sigma(D(\delta, v)) = D'(\delta, \sigma(v))$  (or  $\sigma(\delta v) = \delta(\sigma(v))$ )

# Isomorphisms

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**Proposition** *Let  $G$  be the  $d \times d'$  matrix such that  $\sigma\alpha = G\beta$ . Then*

- 1.  $\sigma$  is a (differential) homomorphism if and only if  $\delta_i G = A_i G - G B_i$ .*
- 2.  $\sigma$  is a (differential) isomorphism if and only if  $d = d'$ ,  $G$  is invertible, and  $A = G(B)$ .*
- 3. If  $d = d'$ , there is a bijection between the set of isomorphisms from  $\mathfrak{V}$  to  $\mathfrak{V}'$  and the set of  $G \in GL(d, \mathfrak{F})$  such that  $A = G(B)$ .*

## Ordinary Case $m = 1$

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- $\mathcal{V} = (V, D)$ , an  $d$ -dimensional  $\delta$ -vector space over  $\mathcal{F}$  with basis  $\alpha : \alpha_1, \dots, \alpha_d$
- Structure matrix for  $\delta$  relative to basis is  $A = (a_{i,j})$  where
$$\delta\alpha_i = -\sum a_{j,i}\alpha_j \text{ or } \delta\alpha = -A^T\alpha.$$
- Earlier, we define the structure matrix as  $-A^T$  rather than  $A$ .
- Let  $v = v_1\alpha_1 + \dots + v_d\alpha_d \in V$
- $\alpha_v = (v_1, \dots, v_d)^T \in \mathcal{F}^d$ .
- Then  $\alpha_v$  satisfies  $\delta Y = AY \iff \delta v = 0$
- $\sigma$  is a differential homomorphism if and only if  $\delta G = BG - GA$ .
- $\sigma$  is a differential isomorphism if and only if  $B = G(A)$ .

# Dual Differential Vector Space

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- $V^*$ , dual vector space of  $V$ , dual basis  $\beta : \beta_1, \dots, \beta_d$  ( $\beta = \alpha^*$ )
- Define  $D^*$  action of  $\Delta$  by

$$(\delta f)(v) = -f(\delta v) + \delta(f(v)) \text{ for } f \in V^*, v \in V$$

- Relative to the dual basis  $\beta$ , the structure matrix of  $V^*$  is  $B = -A^T$  (that is,  $\delta\beta = A\beta$ ,  $\delta\alpha = B\alpha$ ).
- $V^{**}$  is isomorphic to  $V$  and has the same structure matrix  $A$  when  $\alpha^{**}$  is identified with  $\alpha$ .

# Cyclic Vectors

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- $\mathcal{V} = (V, D)$  is *cyclic* if it has a basis  $\alpha$  of the form  $v, \delta v, \dots, \delta^{(d-1)}v$ .
- $v$  cyclic vector implies  $\delta\alpha = B\alpha$  where

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -b_0 & -b_1 & \cdots & \cdots & -b_{d-1} \end{pmatrix}$$

- $\mathcal{V}$  cyclic, then structure matrix (earlier definition) relative to basis  $\alpha$  is companion matrix  $B$  (or  $A = -B^T$ )
- Conversely, if some structure matrix is similar to this form, then  $\mathcal{V}$  is cyclic. If  $\mathfrak{F} \neq \mathfrak{C}$ , every finite dimensional differential vector space is cyclic. (See Churchill-Kovacic's paper).
- Finding a cyclic vector transforms a first order system to a linear homogeneous equation.



# Lax Equation and Lax Pairs

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- $\mathcal{F}$ , an ordinary differential field
- $\mathfrak{a}$  a (prime) differential polynomial ideal in  $\mathcal{R}$  such that  $\mathcal{R}/\mathfrak{a}$  has finite dimension  $d$ .
- $(G, A)$ , two  $d \times d$  matrix over  $\mathcal{F}[y_1, \dots, y_n]$
- $(G, A)$  is a Lax Pair if  $\delta G = [A, G] \bmod \mathfrak{a}$  (Lax Equation)
- A Lax Pair is the same as a differential endomorphism  $\sigma$  on a differential vector space  $\mathcal{V} = \mathcal{R}/\mathfrak{a}$  with a structure matrix  $A$  relative to some basis  $\alpha$ .
- If  $G$  is invertible,  $\sigma$  is an automorphism.

## Example

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- System of ODE given by

$$\delta y_1 = y_1(y_2 - y_3)$$

$$\delta y_2 = y_2(y_3 - y_1)$$

$$\delta y_3 = y_3(y_1 - y_2)$$

- Let  $(\eta_1, \eta_2, \eta_3)$  be a generic solution over  $\mathbb{Q}$
- $\mathcal{V} = \mathbb{Q}\langle \eta_1, \eta_2, \eta_3 \rangle = \mathbb{Q}(\eta_1, \eta_2, \eta_3)$
- A Lax Pair is

$$G = \begin{pmatrix} 0 & 1 & y_1 \\ y_2 & 0 & 1 \\ 1 & y_3 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} y_1 + y_2 & 0 & 1 \\ 1 & y_2 + y_3 & 0 \\ 0 & 1 & y_3 + y_1 \end{pmatrix} \quad (7)$$

## Some Simple Results on Lax Pairs

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- The eigenvalues of  $G$  (evaluated at  $\eta$ ) are  $\delta$ -constants (that is, constants of motion or first integrals)
- The trace and determinant of  $G$  are also first integrals.
- If  $G$  is diagonalizable, then the eigenvalues of all powers of  $G$  and their traces and determinants, are first integrals of  $\mathbf{a}$ .
- In the example,  $\det(G) = y_1 y_2 y_3$  and  $\delta(\eta_1 \eta_2 \eta_3) = 0$ ;
- $\det(G^2) = y_1 + y_2 + y_3$ ;
- the eigenvalues are functions of these.
- There is no known general algorithm for finding Lax pairs for arbitrary system of ODE.

## John Baez's Comments on Lax Pairs

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“Well, for example, the fact that you get an infinite set of conserved quantities for the KdV equations by using the solution (at any given moment in time) as the potential for a Schrodinger-type Hamiltonian and using the eigenvalues of this Hamiltonian as the conserved quantities. In other words, the Lax pair idea. By now this is old news, but if you remember that the KdV equation started out as a description of waves in shallow water, you'll understand how surprised people were to find quantum mechanics sneaking into the game!

It was then surprising how the Lax pair idea wound up being generalized to cover vast classes of integrable systems, and how it wound up being related to the Yang-Baxter equations, quantum groups, and ultimately knot theory.”

# Differential Field Extensions

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- $\mathfrak{G}, \mathfrak{F}$  differential fields
- $\mathfrak{G}$  is a (differential) extension of  $\mathfrak{F}$  if  $\mathfrak{G} \supseteq \mathfrak{F}$  and

$$\delta : \mathfrak{G} \rightarrow \mathfrak{G} \text{ restricts to } \delta : \mathfrak{F} \rightarrow \mathfrak{F}$$

- $\mathfrak{G}$  is a *finitely generated* extension of  $\mathfrak{F}$  if there exist  $\eta_1, \dots, \eta_n \in \mathfrak{G}$  such that

$$\mathfrak{G} = \mathfrak{F}(\{\theta\eta_j\}_{\theta \in \Theta, 1 \leq j \leq n})$$

- Notation:  $\mathfrak{G} = \mathfrak{F}\langle \eta_1, \dots, \eta_n \rangle$

## Example

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- $\mathfrak{G} = \mathbb{Q}\langle e^x, e^{2x} \rangle = \mathbb{Q}(e^x)$
- $y'' - 3y' + 2y = 0$
- $e^x, e^{2x}$  are linearly independent solutions over  $\mathbb{Q}$
- $\mathfrak{G}$  is a Picard-Vessiot extension of  $\mathbb{Q}$ .

# Finitely Generated Extensions and Primes

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- $\mathfrak{p}$  is a prime differential ideal in  $\mathcal{F}\{y_1, \dots, y_n\}$
- $\mathcal{G}$  = quotient field of  $\mathcal{F}\{y_1, \dots, y_n\}/\mathfrak{p}$
- $\eta_i = y_i + \mathfrak{p} \in \mathcal{F}\{y_1, \dots, y_n\}/\mathfrak{p}$
- $\mathcal{G} = \mathcal{F}\langle \eta_1, \dots, \eta_n \rangle$
- $\mathcal{F}\{y_1, \dots, y_n\} \longrightarrow \mathcal{F}\{y_1, \dots, y_n\}/\mathfrak{p} = \mathcal{F}\{\eta_1, \dots, \eta_n\}$  with  $F \mapsto F + \mathfrak{p} = F(\eta_1, \dots, \eta_n)$
- $F(\eta) = 0$  iff  $F \in \mathfrak{p}$
- The set of elements in  $\mathcal{F}\{y_1, \dots, y_n\}$  vanishing at  $\eta = (\eta_1, \dots, \eta_n)$  is a prime differential ideal.
- Finitely generated extensions  $\leftrightarrow$  Prime differential ideals

# Semi-Universal Extension Field

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- $\mathcal{U}$  is *semi-universal* extension of  $\mathcal{F}$  if
  1.  $\mathcal{F}$  is a differential subfield of  $\mathcal{U}$ ,
  2. every finitely generated (differential) extension of  $\mathcal{F}$  may be embedded in  $\mathcal{U}$  (can solve in  $\mathcal{U}$  any PDE system with coefficients in  $\mathcal{F}$ ).
- Every differential field has a separable semi-universal extension field.



# Universal Extension Field

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- $\mathcal{U}$  is *universal* extension of  $\mathcal{F}$  if  $\mathcal{U}$  is semi-universal over every finitely generated extension of  $\mathcal{F}$  (can solve in  $\mathcal{U}$  any PDE system with coefficients in any f.g. extension of  $\mathcal{F}$ ).
- $\mathcal{U}$  is a *universal differential field* if it is universal over its prime field.
- Every differential field has a separable universal extension.
- Universal differential fields are not unique.
- Universal differential fields are analogues of universal fields in classical algebraic geometry.

# Differential Zariski Topology

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- $F \in \mathcal{R} = \mathcal{F}\{y_1, \dots, y_n\}$
- $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{U}^n$
- $\eta$  is a *zero* or *solution* of  $F$  if  $F(\eta) = 0$
- $U \subseteq \mathcal{R}$ , the set of common zeros of  $U$  is

$$\mathcal{Z}(U) = \{ \eta \in \mathcal{U}^n \mid F(\eta) = 0 \text{ for } F \in U \}$$

- The set  $\{ \mathcal{Z}(U) \mid U \subseteq \mathcal{R} \}$  forms the family of closed sets of a differential Zariski  $\mathcal{F}$ -topology (Kolchin  $\mathcal{F}$ -topology).
- The set of elements in  $\mathcal{F}\{y_1, \dots, y_n\}$  vanishing at a point  $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{U}^n$  is a prime differential ideal.
- The set of elements in  $\mathcal{F}\{y_1, \dots, y_n\}$  vanishing at every point  $\eta = (\eta_1, \dots, \eta_n) \in W \subseteq \mathcal{U}^n$  is a radical differential ideal.

# The Logarithmic Derivative Map

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- $G$ , an  $d \times d$  invertible matrix over  $\mathbf{u}$
- The *logarithmic derivative* of  $D$  is  $\ell\delta(G) = \delta G \cdot G^{-1}$ .
- (Kovacic) If  $\mathfrak{G}$  is a connected algebraic subgroup of  $GL(n, \mathbf{u})$  defined over  $\mathcal{C}$ , the  $\ell\delta : \mathfrak{G} \longrightarrow \ell(\mathfrak{G})$  is surjective (where  $\ell(\mathfrak{G})$  is the Lie algebra of  $\mathfrak{G}$ ).
- If  $m = |\Delta| > 1$ , let  $\mathbb{I}$  be the set of  $(A_1, \dots, A_m) \in \ell(\mathfrak{G}) \times \dots \times \ell(\mathfrak{G})$  satisfying the integrability conditions

$$\delta_i A_j - \delta_j A_i = [A_i, A_j]$$

- $\mathbb{I}$  is a  $\Delta$ -subgroup of the additive group  $\ell(\mathfrak{G}) \times \dots \times \ell(\mathfrak{G})$
- $\ell\Delta : G \mapsto (\ell\delta_1 G, \dots, \ell\delta_m G)$  maps  $\mathfrak{G}$  surjectively onto  $\mathbb{I}$ .

# Galois Groups of Linear Homogeneous ODE

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- $\mathcal{F}$ , an ordinary differential field, characteristic zero
- $\mathcal{C}$ , constant subfield of  $\mathcal{F}$ , algebraically closed
- $L = y^{(d)} + a_{d-1}y^{(d-1)} + \cdots + a_0y$ ,  $a_i \in \mathcal{F}$
- $\alpha = (\alpha_1, \dots, \alpha_d)$ , a fundamental system of zeros in  $\mathcal{U}$  such that
- $\mathcal{G} = \mathcal{F}\langle\alpha\rangle$  has field of constant  $\mathcal{C}$
- $\sigma : \mathcal{G} \longrightarrow \mathcal{U}$  an isomorphism over  $\mathcal{F}$
- $\sigma(\alpha_i) = \sum c_{i,j}\alpha_j$
- $\sigma \mapsto (c_{i,j})$  defines a Galois group  $\mathfrak{G}$  contained in  $GL(d, \mathcal{C})$
- Problem: Given  $L$ , compute  $\mathfrak{G}$ .

# Galois Groups of First Order Systems

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- Introduce new differential indeterminates  $Y = (y_1, \dots, y_d)^T$
- $A$ , any  $d \times d$  matrix over  $\mathcal{F}$
- First order system  $Y' = AY$
- $\eta_j = (\eta_{1,j}, \dots, \eta_{d,j})^T$  are linearly independent solutions
- $\eta = (\eta_{i,j})$ ,  $\mathcal{G} = \mathcal{F}\langle\eta\rangle = \mathcal{F}(\eta)$  is Picard Vessiot if field of constants of  $\mathcal{G}$  is  $\mathbb{C}$ .
- $\sigma : \mathcal{G} \longrightarrow \mathcal{U}$  an isomorphism over  $\mathcal{F}$
- $\sigma(\eta) = C\eta$ ,  $C = (c_{i,j}) \in GL(d, \mathbb{C})$
- $\sigma \mapsto (c_{i,j})$  defines a Galois group  $\mathfrak{G}$  contained in  $GL(d, \mathbb{C})$

## Special Case

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- $L = y^{(d)} + a_{d-1}y^{(d-1)} + \dots + a_0y$ ,  $a_i \in \mathcal{F}$
- $\alpha = (\alpha_1, \dots, \alpha_d)$ , a fundamental system of zeros in  $\mathbf{u}$  such that
- $\mathcal{G} = \mathcal{F}\langle\alpha\rangle$  has field of constant  $\mathcal{C}$
- Rewrite  $L$  as a first order system:  $Y' = AY$ , where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & \cdots & \cdots & -a_{d-1} \end{pmatrix}$$

- $A$  is called a *companion matrix*.
- $\eta_{i,j} = \alpha_i^{(j)}$ ;  $\eta = (\eta_{i,j})$  is a solution to  $Z' = AZ$  where  $Z = (z_{i,j})$ .
- $\eta_j = (\eta_{1,j}, \dots, \eta_{d,j})^T$  are linearly independent zeros of  $Y' = AY$ .
- $\mathcal{G} = \mathcal{F}(\{\eta_{i,j}\}_{1 \leq i,j \leq d})$  is Picard Vessiot.

# Differential Vector Spaces over $\mathfrak{u}$

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**Theorem** Let  $\mathfrak{u}$  be a universal differential field. Then any two  $d$ -dimensional differential vector space over  $\mathfrak{u}$  are isomorphic.

- $\mathfrak{V} = (\mathfrak{u}^d, D)$  the natural differential vector space where  $D(\delta, x) = (\delta x_1, \dots, \delta x_d)$ .
- Let  $\alpha : \alpha_1, \dots, \alpha_d$  be the canonical basis.
- The representation of  $D$  is then  $A = (A_1, \dots, A_m)$ , where  $A_i = 0$  for all  $i$  ( $\alpha_j$  has constant coordinates)
- For any  $G \in GL(d, \mathfrak{u})$ ,  $G(A) = \ell\Delta(G)$ .
- The image of  $\ell\Delta$  is  $\mathbb{I} \subset gl(d, \mathfrak{u})^m$ .
- $\mathcal{E}^d(\mathfrak{u})$  consists of singleton  $\prec A \succ$
- $\mathcal{D}^d(\mathfrak{u})$  is a singleton.

# Differential Variety of Linear Differential Ideals

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**Theorem** If  $\mathcal{V}$  is any  $d$ -dimensional  $\mathcal{K}$ -subspace of  $\mathcal{U}^n$ , then  $\mathcal{V}$  is differential Zariski closed, and its defining differential ideal is linear of linear dimension  $d$ . Conversely, the zero set of any linear differential ideal  $\mathfrak{p}$  of linear dimension  $d$  is a  $d$  dimensional  $\mathcal{K}$ -subspace of  $\mathcal{U}^n$ .

- The above associations are bijective and inverse of each other.
- $\mathfrak{p}$  has a fundamental system of zeros  $\alpha : \alpha_1, \dots, \alpha_d \in \mathcal{U}^n$ .
- The field of constants of  $\mathfrak{G} = \mathcal{F}\langle\alpha\rangle$  is algebraic over  $\mathcal{C}$ .
- There exists  $Q_1, \dots, Q_d \in \mathcal{U}\{y_1, \dots, y_n\}$  such that
  1. The map  $eval : v \mapsto (Q_1(v), \dots, Q_d(v))$  for  $v \in \mathcal{V}$  is a  $\mathcal{K}$ -vector space  $\Delta$ - $\mathcal{U}$ -isomorphism from  $\mathcal{V}$  into  $\mathcal{K}^d$ .
  2.  $Q_1, \dots, Q_d$  together involve exactly  $d$  distinct derivatives of  $y_1, \dots, y_n$ .



# Lie Vector Spaces

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- $\mathcal{U}$  a universal differential field
- $\mathcal{D}$ , the vector space over  $\mathcal{U}$  generated by  $\Delta$ .
- $D \in \mathcal{D}$  acts as derivation on  $\mathcal{U}$ .
- $\mathcal{K}(D) = \text{set of } u \in \mathcal{U} \text{ with } Du = 0$
- A subspace  $\mathcal{E}$  is a *Lie subspace* if
  - $Du = 0$  for every  $u \in \cap_{E \in \mathcal{E}} \mathcal{K}(E)$  implies  $D \in \mathcal{E}$
  - $\mathcal{E}$  has a basis  $E_1, \dots, E_e$  that commutes.
- (Cassidy) The two conditions are equivalent.

# Field of constants of linear differential ideals

---

- $\mathcal{U}$  universal differential field over  $\mathcal{F}$
- $\mathfrak{p} \subset \mathcal{F}\{y_1, \dots, y_n\}$  linear differential ideal
- $L : L_1, \dots, L_k$  canonical characteristic set of  $\mathfrak{p}$
- $\mathcal{V}$  the set of solutions of  $\mathfrak{p}$  in  $\mathcal{U}^n$
- $\mathcal{V}$  is a differential algebraic subgroup of  $\mathbf{G}_a^n$  ( $\Delta$ - $\mathcal{F}$ -group)
- $\mathcal{K}(\mathcal{V}) =$  set of all  $a \in \mathcal{U}$  with  $a\mathcal{V} \subset \mathcal{V}$
- (Sit) There exists a unique Lie subspace  $\mathcal{E}$  such that  $\mathcal{K}(\mathcal{V}) = \mathcal{K}(\mathcal{E})$ .

# Properties

---

- $L_j(ay_1, \dots, ay_n) = aL_j(y_1, \dots, y_n)$  for all  $1 \leq j \leq k, a \in \mathcal{K}(\mathcal{V})$
- $\mathcal{K}(\mathcal{V})$  is algebraically closed, contains  $\mathcal{K}$
- $\mathcal{V}$  is a vector space over  $\mathcal{K}(\mathcal{V})$
- $\mathcal{K}(\mathcal{V})$  is an infinite differential algebraic subgroup of  $\mathbf{G}_a$
- $\mathcal{K}(\mathcal{V})^*$  is an infinite differential algebraic subgroup of  $\mathbf{G}_m$ .

# Differential Lie Algebras

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- $sl(d, \mathcal{F})$  is a  $\delta$ -Lie algebra over  $\mathcal{C}$
- $A \in sl(d, \mathcal{F})$ ,  $d \times d$  matrix over  $\mathcal{F}$  with trace zero
- $\mathfrak{g}_A = \{ G \in sl(d, \mathcal{F}) \mid \delta G = [A, G] \}$
- $\mathfrak{g}_A$  is a proper, Zariski-dense,  $\delta$ -Lie subalgebra over  $\mathcal{C}$  of  $sl(d, \mathcal{F})$
- Conversely, every proper, Zariski-dense,  $\Delta$ -Lie subalgebra  $\mathfrak{g}$  over  $\mathcal{C}$  of  $sl(d, \mathcal{F})$  is some  $\mathfrak{g}_A$ .
- Such  $\mathfrak{g}$  is conjugate to  $sl(d, \mathcal{C})$  by a matrix in  $SL(d, \mathcal{F})$

# Zariski-dense $\Delta$ -Lie Subalgebras

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- $\mathfrak{G}$ , a connected simple  $\mathbb{C}$ -subgroup of  $GL(d, \mathfrak{F})$
- $\mathfrak{g} = \ell(\mathfrak{G})$ , the Lie algebra of  $\mathfrak{G}$
- $\mathcal{D}$ , vector space over  $\mathfrak{F}$  with basis  $\Delta$
- $\mathcal{D}$  is a Lie space, and a  $\Delta$ -Lie algebra over  $\mathbb{C}$
- $\mathfrak{h}$ , a  $\Delta$ -Lie subalgebra of  $\mathfrak{g}$  over  $\mathbb{C}$
- Then  $\mathfrak{h}$  is Zarisk-dense in  $\mathfrak{g}$  if and only if there exist  $\delta'_1, \dots, \delta'_{m'} \in \mathcal{D}$ , linearly independent over  $\mathfrak{F}$  and commuting, and  $A_1, \dots, A_{m'} \in \mathbb{I}$  such that  $\mathfrak{h}$  is the set of matrices  $G \in \mathfrak{g}$  satisfying

$$\delta'_i G = [A_i, G], i = 1, \dots, m'$$

# Intermission

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## Linear Differential Ideals, II

# Differential Algebraic Dependence

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- $U$  universal differential extension over  $\mathcal{F}$
- $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{U}^n$
- $\eta$  is  $\Delta$ -algebraically dependent over  $\mathcal{F}$  if  $\{\theta\eta_j\}_{\theta \in \Theta, 1 \leq j \leq n}$  is algebraic dependent over  $\mathcal{F}$
- If not, say  $\eta$  is  $\Delta$ -algebraically independent over  $\mathcal{F}$ .
- $(\eta_1, \eta_2) = (\tan x, \sin x)$  is  $\Delta$ -algebraically dependent over  $\mathbb{Q}$  since  $\delta(\tan x)(1 - \sin^2 x) = 1$ .
- $(\eta_1, \eta_2) = (x, J_n(x))$  is  $\Delta$ -algebraically dependent over  $\mathbb{Q}$ .

$$\eta_1^2 \delta^2 \eta_2 + \eta_1 \delta \eta_2 + (\eta_1^2 - n^2) \eta_2 = 0.$$

# Differentially Algebraic Elements

---

- $\alpha \in \mathcal{U}$  is  $\Delta$ -algebraic over  $\mathcal{F}$  if it satisfies some differential polynomial equation with coefficients in  $\mathcal{F}$ .
- If not, say  $\alpha$  is  $\Delta$ -transcendental over  $\mathcal{F}$ .
- $e^x$ ,  $\sin x$ , or  $\cos x$  is transcendental (not algebraic), but  $\Delta$ -algebraic over  $\mathbb{Q}$ .
- $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$  is  $\Delta$ -transcendental (not  $\Delta$ -algebraic) over  $\mathbb{C}(x)$ .
- $J_n(x)$  is  $\Delta$ -algebraic over  $\mathbb{Q}(x)$ .



# Differential Transcendence Basis

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Let  $\mathfrak{G}$  be an extension of  $\mathfrak{F}$ ,  $\Sigma \subset \mathfrak{G}$ . The following are equivalent:

1.  $\Sigma$  is  $\Delta$ -algebraically independent over  $\mathfrak{F}$ , and  $\mathfrak{G}$  is  $\Delta$ -algebraic over  $\mathfrak{F}\langle\Sigma\rangle$ .
2.  $\Sigma$  is a minimal such that  $\mathfrak{G}$  is  $\Delta$ -algebraic over  $\mathfrak{F}\langle\Sigma\rangle$ .
3.  $\Sigma$  is a maximal that is  $\Delta$ -algebraically independent over  $\mathfrak{F}$ .
  - Such  $\Sigma$  is a  $\Delta$ -*transcendence basis* of  $\mathfrak{G}$  over  $\mathfrak{F}$ .
  - $\Delta$ -transcendence basis exists.
  - Any two have the same cardinal number, called the  $\Delta$ -dimension (or  $\Delta$ -transcendence degree) of  $\mathfrak{G}$  over  $\mathfrak{F}$ .
  - $\Delta\text{-dim } \mathcal{H}/\mathfrak{F} = \Delta\text{-dim } \mathcal{H}/\mathfrak{G} + \Delta\text{-dim } \mathfrak{G}/\mathfrak{F}$

## Generic zeros

---

- $\mathfrak{p} = \{ F_1, \dots, F_p \}$  a prime differential ideal in  $\mathcal{F}\{ y_1, \dots, y_n \}$
- $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{U}^n$  is a *generic* zero of  $\mathfrak{p}$  if the set of all differential polynomials vanishing at  $\eta$  is  $\mathfrak{p}$ .

- The  $\Delta$ -dimension of  $\mathcal{F}\langle \eta \rangle$  is the  $\Delta$ -dimension of  $\mathfrak{p}$  which measures the arbitrariness of the solutions of the system

$$F_1 = 0, \dots, F_p = 0.$$

- A finer measurement is the transcendence degree  $\omega(s)$  of the field extension

$$\mathcal{F}(\{ \theta \eta_j \}_{\theta \in \Theta, 1 \leq j \leq n, \text{ord } \theta \leq s})$$

for all  $s$ .

- $\omega(s)$  is given by a polynomial (Kolchin differential dimension polynomial) when  $s$  is large.

# Lattice points and Initial Sets

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- $m > 0$ ,  $\mathbb{N}^m$  with product order  $\leq$
- $V \subseteq \mathbb{N}^m$  is *initial* if  $v \in V, u \in \mathbb{N}^m$  and  $u \leq v$  implies  $u \in V$
- staircase, Ferrer diagram, order ideal of monomials
- $u \in \mathbb{N}^m$ , call  $u + \mathbb{N}^m$  the  $m$ -dim cone based at  $u$  (T-staircase).
- Let  $E(V)$  be the set of minimal elements in  $\mathbb{N}^m \setminus V$  (minimal cogenerators).  
 $E(V)$  is finite.
- $E(V)$  generates the complement of  $V$  as a finite union of cones based at  $e \in E(V)$ .
- Conversely, given a finite set  $E \subset \mathbb{N}^m$ , the complement  $V(E)$  of the union of cones at  $e \in E$  is initial.
- $V(E)$  consists of all points  $v \in \mathbb{N}^m$  that are not greater than or equal to any point in  $E$ .

# Counting Lattice Points

---

- $v = (v_1, \dots, v_m)$ ,  $|v| = v_1 + \dots + v_m$
- $V(s) = \{ v \in V \mid |v| \leq s \}$

**Proposition** *Given  $V$  initial (or  $E(V)$  finite) subset of  $\mathbb{N}^m$ , there exists a numerical polynomial  $\omega_V(X)$  ( $\omega_E$ ) such that for all sufficiently large  $s$ ,  $\omega_V(s) = \text{Card}(V(s))$ .*

- $\deg \omega_V \leq m$
- $\deg \omega_V = m \iff V = \mathbb{N}^m$  (and  $\omega_V = \binom{X+m}{m}$ )
- $\omega_V = 0 \iff V = \emptyset$

## Application: Polynomial Ideals

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- Associate the exponents of a monomial  $X_1^{e_1} \cdots X_m^{e_m}$  with a lattice point  $(e_1, \dots, e_m)$ .
- $I$  an ideal in the polynomial ring  $k[X_1, \dots, X_m]$  (Fix a term ordering).
- $V$  corresponds to the set of monomials that are not leading monomials of a polynomial in  $I$ .
- $E(V)$  corresponds to the leading monomials of a reduced Gröbner basis.
- The monomials corresponding to  $V(s)$  are linearly independent over  $k \bmod I$ .
- $\omega_V(X)$  is the Hilbert polynomial for  $k[X_1, \dots, X_m]/I$ .

## Application: Prime Differential Ideals

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- Associate the orders of a derivative  $\delta_1^{e_1} \cdots \delta_m^{e_m} y$  with a lattice point  $(e_1, \dots, e_m)$ .
- $\mathfrak{p}$ , a linear differential ideal in  $\mathcal{F}\{y\}$  (Fix a ranking).
- $V$  corresponds to the set of derivatives that are not leaders of a linear differential polynomial in  $\mathfrak{p}$ .
- $E(V)$  corresponds to the leaders of a linear characteristic set.
- The derivatives corresponding to  $V(s)$  are algebraically independent over  $\mathcal{F} \bmod \mathfrak{p}$ .
- $\omega_V(X)$  is the Kolchin dimension polynomial for  $\mathcal{F}\langle\eta\rangle$  where  $\eta$  is a generic zero of  $\mathfrak{p}$ .

# Kolchin Differential Dimension Polynomial

---

Let  $\eta = (\eta_1, \dots, \eta_n)$  be a finite family of elements of an extension of  $\mathcal{F}$ . There exists a numerical polynomial  $\omega = \omega_{\eta/\mathcal{F}}$  with the following properties:

1. For every large  $s \in \mathbb{N}$ , the transcendence degree of

$\mathcal{F}(\{\theta\eta_j\}_{\theta \in \Theta, 1 \leq j \leq n, \text{ord } \theta \leq s})$  over  $\mathcal{F}$  equals  $\omega(s)$ .

2.  $\deg \omega \leq m$ .

3. If we write  $\omega(X) = \sum_{i=0}^m a_i \binom{X+i}{i}$ , then  $a_m = \Delta\text{-dim } \mathcal{F}(\mathcal{F}\langle\eta\rangle)$ .

# Computation of Kolchin Polynomial

---

- $\mathfrak{p} = \{F \in \mathcal{F}\{y_1, \dots, y_n\} \mid F(\eta) = 0\}$ , the *defining (prime) differential ideal* of  $\eta$  over  $\mathcal{F}$
- $\mathbf{A}$ , a characteristic set of  $\mathfrak{p}$  relative to an *orderly* ranking.
- For each  $j$  ( $1 \leq j \leq n$ ), let  $E_j$  be the set of points  $(e_1, \dots, e_m)$  such that  $\delta_1^{e_1} \cdots \delta_m^{e_m} y_j$  is a leader of an element of  $\mathbf{A}$ .

- Then

$$\omega = \sum_{j=1}^m \omega_{V(E_j)}.$$



## Examples (Linear Differential Ideals)

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- $m = 2, n = 2, h > 0$ , orderly ranking
- $P_{1,1} = \delta_1^h y_1$
- $P_{1,2} = \delta_1 \delta_2^{h-1} y_2 + y_1$
- $I_1 = [P_{1,1}, P_{1,2}]$
- $\mathbf{A}_1 = \{ P_{1,1}, P_{1,2} \}$  is a characteristic set of  $I_1$ .
- $P_{2,1} = \delta_2 \delta_1^{h-1} y_1 + y_2$
- $P_{2,2} = \delta_2^h y_2$
- $I = [P_{1,1}, P_{1,2}, P_{2,1}, P_{2,2}]$
- $\mathbf{A} = \{ y_1, y_2 \}$  is a characteristic set of  $I$ .

## Example

---

- The Kolchin dimension polynomials are:

$$\omega_{I_1}(X) = \omega_{I_2} = 2 \left[ \binom{X+2}{2} - \binom{X-h+2}{2} \right] = 2h \binom{X+1}{1} + (h-h^2),$$

and  $\omega_I(X) = 0$ .

- The types for  $I_1$  and  $I_2$  are both 1 and the type of  $I$  is zero.
- The typical dimensions of  $I_1$  and  $I_2$  are  $2h$ , and the typical dimension of  $I$  is 0.
- The general solution of  $I_1$  or of  $I_2$  depends on  $h$  arbitrary functions of one independent variable (after a transformation of the independent variables).
- This example also shows that two rather large differential algebraic varieties can intersect at a only one point.

# Dimension Polynomials

---

**dimension polynomial:** A numerical polynomial  $\omega(X)$  which can be written as a finite sum of  $\omega_{V_i}(X)$  where each  $V_i$  is an initial set in  $\mathbb{N}^m$ .

**Notation:**  $\omega(X) = \sum_{i=0}^m a_i \binom{X+i}{i}, \quad a_m \geq 0$

**Examples:** Hilbert Polynomial, Kolchin Polynomial

**ordering:** Say  $\omega(X) \leq \omega'(X)$  if  $\omega(s) \leq \omega'(s)$  for all big  $s$ .

$$\omega \leq \omega' \iff (a_m, \dots, a_0) \leq_{lex} (a'_m, \dots, a'_0)$$

# Properties

---

## Well-ordering Theorem:

Dimension polynomials are well-ordered under  $\leq$ .

**characterisation:** A numerical polynomial  $\omega(X) = \sum_{i=0}^d a_i \binom{X+i}{i}$  with  $a_d \neq 0$  is a dimension polynomial if and only if  $a_d > 0$  and

$$\omega(X + a_d) - \binom{X + d + 1 + a_d}{d + 1} + \binom{X + d + 1}{d + 1}$$

(degree  $\leq d - 1$ ) is a dimension polynomial.

# Effects of Change of Generators

---

- $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{U}^n$
- $\zeta = (\zeta_1, \dots, \zeta_{n'}) \in \mathcal{U}^{n'}$
- If  $\eta_j \in \mathcal{F}(\{\theta\zeta_k\}_{\theta \in \Theta(h), 1 \leq k \leq n'})$  for  $1 \leq j \leq n$ , then

$$\mathcal{F}(\{\theta\eta_k\}_{\theta \in \Theta(s), 1 \leq k \leq n}) \subseteq \mathcal{F}(\{\theta\zeta_k\}_{\theta \in \Theta(s+h), 1 \leq k \leq n'})$$

$$\omega_{\eta/\mathcal{F}}(X) \leq \omega_{\zeta/\mathcal{F}}(X + h).$$

- If  $\mathcal{F}\langle\eta\rangle = \mathcal{F}\langle\zeta\rangle$ , then there exists  $h \in \mathbb{N}$  such that

$$\omega_{\zeta/\mathcal{F}}(X - h) \leq \omega_{\eta/\mathcal{F}}(X) \leq \omega_{\zeta/\mathcal{F}}(X + h).$$

## Birational and $\Delta$ -birational Invariants

---

- If  $\mathcal{F}(\eta) = \mathcal{F}(\zeta)$  ( $h = 0$ ), then  $\omega_{\eta/\mathcal{F}} = \omega_{\zeta/\mathcal{F}}$ .
- $\omega_{\eta/\mathcal{F}}$  is a birational invariant, but not differential birational invariant.
- $\tau = \deg \omega_{\eta/\mathcal{F}}$  and leading coefficient  $a_\tau$  are differential birational invariants.
- $\tau$  is the *differential type* of  $\mathcal{F}\langle\eta\rangle$  over  $\mathcal{F}$
- $a_\tau$  is the *typical differential dimension*.

## Transformation of $\Delta$

---

- $\mathcal{C}$  is the field of constants of  $\mathcal{F}$ .
- $C = (c_{i,i'})_{1 \leq i, i' \leq m} \in GL(m, \mathcal{C})$
- $\delta_i = \sum_{i'=1}^m c_{i,i'} \delta'_{i'}$
- $\Delta' = \{ \delta'_1, \dots, \delta'_m \}$
- May view  $\mathcal{F}$  as a  $\Delta'$ -field.

**Theorem** *Let  $\mathcal{G}$  be a finitely generated extension of  $\mathcal{F}$ . Let  $\tau$  be the differential type and  $a_\tau$  the typical differential dimension of  $\mathcal{G}$  over  $\mathcal{F}$ . There there exists a set  $\Delta'$  of  $\tau$  linearly independent elements of  $\sum_{\delta \in \Delta} \mathcal{C} \cdot \delta$  such that  $\mathcal{G}$  is a finitely generated  $\Delta'$  extension of  $\mathcal{F}$  of  $\Delta'$ -dimension  $a_\tau$ .*

**Remark** The  $\tau \times m$  matrix over  $\mathcal{C}$  that gives  $\Delta'$  may be chosen from a Zariski open set.

# Invariant Properties and Meaning

---

- $\tau$  is the maximum number of independent variables on which depends a general solution of

$$F_1 = 0, \dots, F_p = 0.$$

- $a_\tau$  is the number of arbitrary functions of  $\tau$  variables on which depends a general solution of

$$F_1 = 0, \dots, F_p = 0.$$

- The differential type  $\tau$  and typical differential dimension  $\mathbf{a}_\tau$  are invariant under transformation of the independent variables ( $\Delta$  to  $\Delta'$ ) and transformation of the dependent variables ( $\mathcal{F}\langle\eta\rangle = \mathcal{F}\langle\zeta\rangle$ )



# Zero Sets of Linear Differential Ideals

---

- $\mathcal{K}$  = constant field of  $\mathcal{U}$
- Set of zeros is a vector space  $V(\mathfrak{p})$  of  $\mathcal{U}^n$  over  $\mathcal{K}$ .
- $V(\mathfrak{p})$  is a differential algebraic subgroup of  $\mathbf{G}_a^n$ .
- Conversely, a differential algebraic subgroup of  $\mathbf{G}_a^n$  is defined by a linear differential ideal.
- $\omega_{\mathfrak{p}}(s) = \dim \mathcal{L}(s) / (\mathfrak{p} \cap \mathcal{L}(s))$

## Intersections in $\mathbf{G}_a^n$

---

- $V_1, V_2$ , two differential algebraic subgroups of  $\mathbf{G}_a^n$ ,
- differential types  $\tau_1, \tau_2$
- Then the differential types of  $V_1 \cap V_2$  and  $V_1 + V_2$  are both  $\leq \tau = \max(\tau_1, \tau_2)$ .
- Moreover,

$$a_\tau(V_1) + a_\tau(V_2) = a_\tau(V_1 \cap V_2) + a_\tau(V_1 + V_2).$$

- $\mathfrak{p}_1$  defining differential ideal of  $V_1$
- $\mathfrak{p}_2$  defining differential ideal of  $V_2$
- $\mathfrak{p}_1 + \mathfrak{p}_2$  defining differential ideal of  $V_1 \cap V_2$
- $\mathfrak{p}_1 \cap \mathfrak{p}_2$  defining differential ideal of  $V_1 + V_2$
- $a_m(V_1 \cap V_2) \geq a_m(V_1) + a_m(V_2) - n$

# Ritt's Example

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- Intersection Theorem does not hold for general Kolchin-closed sets.
- $\mathcal{R} = \mathcal{F}\{u, v, y\}$ ,  $m = 1$
- $F_1 = u^5 - v^5 + y(u\delta v - v\delta u)^2$
- $F_1$  is irreducible.
- $A_j = u - w^{i-1}v$ ,  $[A_j]$  is a component of  $F$ .
- $\{F_1\}$  has 6 components,  $\Delta\text{-dim} = 2$ .
- $F_2 = y$ ,  $\Delta\text{-dim} = 2$ .
- $V(F_1) \cap V(F_2) = \{(0, 0, 0)\}$ .

# Linear Differential Algebraic Groups and Lie Algebras

---

- $G$  differential algebraic subgroup of  $\mathbf{GL}(n, \mathcal{U})$
- All components of  $G$  have same differential dimension polynomials.
- The Lie algebra  $\mathbf{l}(G)$  of  $G$  is a differential algebraic  $\mathcal{K}$ -subalgebra of  $\mathfrak{gl}(n, \mathcal{U})$ .
- $\mathbf{l}(G)$  has same differential dimension polynomial as  $G$ .

## Intersections in $\mathbf{GL}(n, \mathfrak{U})$

---

**Theorem** *Let  $H_1, H_2$  be differential algebraic subgroups of  $\mathbf{GL}(n, \mathfrak{U})$ , of differential types  $\tau_1, \tau_2$  respectively. Then*

$$a_m(H_1 \cap H_2) \geq a_m(H_1) + a_m(H_2) - n^2.$$

*If furthermore  $H_1$  normalizes  $H_2$ , then*

- 1.  $H_1H_2$  is a differential algebraic subgroup of  $\mathbf{GL}(n, \mathfrak{U})$ ,*
- 2. differential types of  $H_1 \cap H_2$  and  $H_1H_2$  are  $\leq \tau = \max(\tau_1, \tau_2)$ ,*
- 3.  $a_\tau(H_1) + a_\tau(H_2) = a_\tau(H_1 \cap H_2) + a_\tau(H_1H_2)$ .*