The Kolchin Seminar in Differential Algebra
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PART I

Differential equations integrable by algebraic geometry: A survey of methods

1992 National Mathematics Awareness Week
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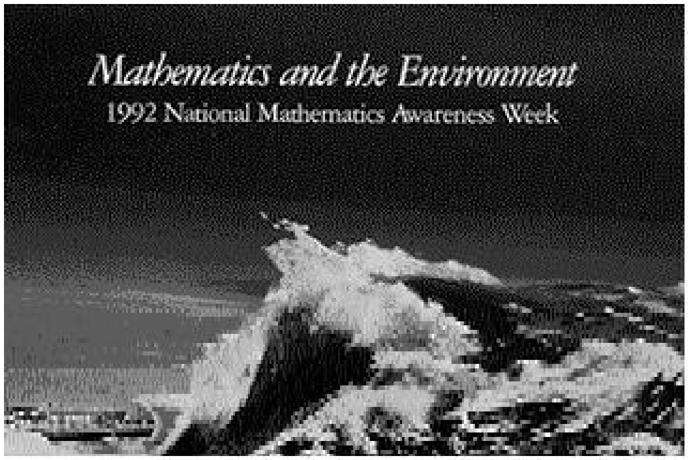


MAW 92 Theme Postcard

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Korteweg-de Vries

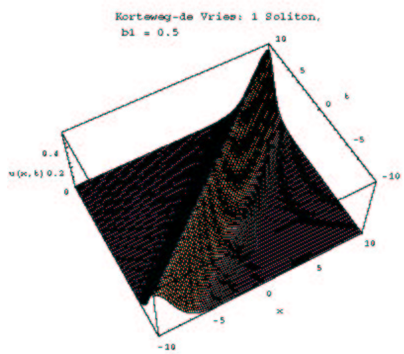
Alex

http://math.cofc.edu/faculty/kasman/SOLITONPICS/hmsol.html

By linear change of variables, rewrite KdV:

$$u_t - 6uu_x + u_{xxx} = 0$$

For a wave equation, it is natural to look for 'wave' solutions, e.g. 1-wave of velocity c so the shape is preserved on $x - ct = \text{const.}$



Picture credit:
`http://www.usf.uni-osnabrueck.de/~ kbrauer/solitons.html`

$$\text{if } u(x,t) = u(x - ct)$$

$$-cu' - 6uu' + u''' = 0$$

$$(-cu - 3u^2 + u'' + a)u' = 0$$

$$\frac{(u')^2}{2} = u^3 + c\frac{u^2}{2} - au + b$$

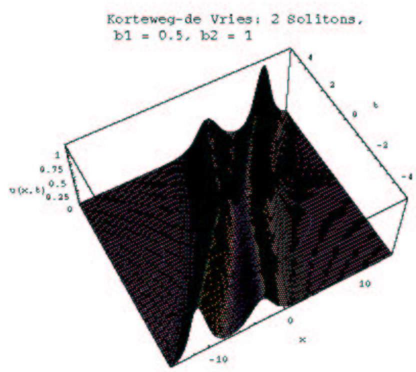
$$u = 2\wp + \text{const.}$$

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

where $\wp$ is the Weierstrass $\wp$ function

$$\wp(z) = \frac{1}{z^2} + \sum_{(n,m)\neq(0,0)} \left(\frac{1}{(z + n\omega_1 + m\omega_2)^2} - \frac{1}{(n\omega_1 + m\omega_2)^2} \right)$$

Two 1-wave solutions will interact in a non-linear way. Experiments in the late 1960s (Gardner, Greene, Kruskal, Miura, Su, Zabusky) showed that shape and velocity were preserved after interaction.



Picture credit: `http://www.usf.uni-osnabrueck.de/~kbrauer`

What emerged was an addition law on the Jacobian of a a spectral curve.

What made it work?

$$z = \frac{1}{2}\int_{\infty}^{\wp(z)} \frac{dw}{\sqrt{(w-e_1)(w-e_2)(w-e_3)}}$$

$$\int_{\infty}^{\wp(z_1)} + \int_{\infty}^{\wp(z_2)} = \int_{\infty}^{\wp(z_1+z_2)}$$

$$\wp(z_1+z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\left[\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right]^2$$

Features of $\wp$:

• $\wp$, $\wp'$ satisfy an algebraic equation (proof based on local expansion around the pole)

• $\wp$ uniformizes the elliptic curve

$$X: \ \mu^2 = 4\lambda^3 - g_2\lambda - g_3$$

i.e., $\mathbb{C}/(\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}) \to X$, $z \mapsto (\lambda = \wp(z), \mu = \wp'(z))$ is an isomorphism of complex varieties,
$\mathbb{C}(X) = \mathbb{C}(\lambda, \mu) \cong \mathbb{C}(\wp(z), \wp'(z))$.

To generalize this to g-variables, we need instead g points on a Riemann surface X of genus g. More precisely (classical):

Choose a normalized basis of $H_1(X,\mathbb{Z})$, A-cycles $A_1, ..., A_g$, B-cycles $B_1, ..., B_g$

Abel map: normalized basis of holomorphic differentials $\omega_1, \ldots, \omega_g$:

$$\int_{A_j} \omega_i = \delta_{ij}; \ \int_{B_j} \omega_i = \tau_{ij}. \ \text{Let } T = [\tau_{ij}] \text{ period matrix.}$$

The period lattice Λ is spanned over $\mathbb{Z}$ by the columns of the $g \times g$ identity matrix and of the period matrix T.

$$A\colon\ X^{(d)} \mapsto \mathbb{C}^g/\Lambda = \mathrm{Jac}(X)$$

$$P_1 + \ldots + P_d \mapsto \sum_{i=1}^{d} \int_{P_0}^{P_i} \vec{\omega},\ P_0 = \text{basepoint},\ \vec{\omega} = (\omega_1, \ldots, \omega_g).$$

Addition law:

$$\sum_{i=1}^{g} \int_{a_0}^{a_i} \omega + \sum_{i=1}^{g} \int_{a_0}^{b_i} \omega \equiv \sum_{i=1}^{g} \int_{a_0}^{c_i} \omega \ (\text{mod periods})$$

Riemann's theta function: $\vec{z} \in \mathbb{C}^g$,

$$\vartheta(\vec{z}) = \sum_{\vec{n} \in \mathbb{Z}^g} \exp(2\pi i {}^t\vec{n} T \vec{n} + \pi i {}^t\vec{n}\vec{z}) \ (\text{quasi periodic w.r.t. } \Lambda)$$

e.g. in $g = 1, \partial_z^2 \log \vartheta \propto -\wp(z)$.

But for a closer generalization:

$$X : \mu^2 = \lambda^{2g+1} + a_{2g}\lambda^{2g} + \cdots + a_0 \quad \text{(hyperelliptic)}$$

H.F. Baker (1903)

$$\omega_1 = \frac{d\lambda}{2\mu},\ \omega_2 = \frac{\lambda d\lambda}{2\mu}, \ldots, \omega_g = \frac{\lambda^{g-1} d\lambda}{2\mu}$$

$$\eta_j = \frac{1}{2\mu} \sum_{k=j}^{2g-j} (k+1-j) a_{k+1+j} \lambda^k d\lambda$$

$$\sigma(z) = \exp\left(-\frac{1}{2}{}^t z H' \Omega'^{-1} z\right) \vartheta \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} (\Omega'^{-1} z; T)$$

$$T = \Omega'^{-1} \Omega''$$

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z) =$$

$$\sum_{n \in \mathbb{Z}^g} \exp\left[2\pi i \left(\frac{1}{2}{}^t(n+a)T(n+a) + {}^t(n+a)(z+b)\right)\right]$$

$$\sigma \propto \vartheta$$

$$\wp_{ij}(z) = -\frac{\partial^2}{\partial z_i \partial z_j} \log \sigma(z)$$

and if $\mu^2 = \prod_{i=1}^{5}(\lambda - \lambda_i)$,

$$\begin{bmatrix} -a_0 & \frac{1}{2}a_1 & 2\wp_{11} & -2\wp_{12} \\ \frac{1}{2}a_1 & -(a_2 + 4\wp_{11}) & \frac{1}{2}a_3 + 2\wp_{12} & 2\wp_{22} \\ 2\wp_{11} & \frac{1}{2}\lambda_3 + 2\wp_{12} & -(\lambda_4 + 4\wp_{22}) & 2 \\ -2\wp_{12} & 2\wp_{22} & 2 & 0 \end{bmatrix}$$

its determinant defines the Kummer surface!

Modern point of view: let L be the line bundle over $A = \mathbb{C}^g/\Lambda_T$ corresponding to the theta divisor Θ.

The 2^g functions

$$\vartheta \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} (z; T), \;\; \delta', \delta'' \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$$

give a basis of $V = H^0(A, L^{\otimes 2})$ and the
Kummer variety $K \cong A/\pm$ is defined by
$\phi_{|2\Theta|} : A \to V^* = \mathbb{P}^{2^g - 1}$.

Study $\Gamma_{[n]\cdot 0} := \{s \in V | \mathrm{mult}_0 s \geq 2n\}$
in particular $\Gamma_{00} := \{s \in V | \mathrm{mult}_0 s \geq 4\}$
(even functions,
$\mathbb{P}\Gamma_{[1]\cdot 0}$ a hyperplane,
$\dim \Gamma_{00} = 2^g - \frac{g(g+1)}{2} - 1$
Schottky problem)

What about KdV? Baker also proved,

$$g = 2: \ (D_3^4 - 4e_6 D_3^2 - 4D_3 D_2 - 4e_5 e_7)\sigma \cdot \sigma = 0$$

where Hirota's bilinear operator is defined as:

$$D_{t_n} F \cdot F = \left(\frac{\partial}{\partial t_n'} - \frac{\partial}{\partial t_n} \right) F(\underline{t}) F(\underline{t}') \Big|_{\underline{t}=\underline{t}'},$$

$$\underline{t} = (t_1, t_2, \ldots)$$

and Hirota's Direct Method gives:

$$\text{KdV} \Leftrightarrow \ (D_x D_t + D_x^4) F \cdot F = 0$$

for $u = 2\dfrac{\partial^2}{\partial x^2} \log F$

Now, what about $g > 2$?

Let's reconsider the reason why KdV had an 'algebraic' nature. Why the curve?

The KdV equation was found (Lax, 1968) to be equivalent to the "Lax-pair" equation:

$$\partial_t L = [B, L]$$

an isospectral deformation of the Schrödinger operator
$L = -\frac{d^2}{dx^2} - u(x,t)$ (t is the deformation parameter)
with $B = -4\partial^3 - 3(u\partial + \partial u)$ (now $\partial = \frac{\partial}{\partial x}$).
Letting $B = U_t U^{-1}$ (where defined) gives $L(t) = UL(0)U^{-1}$:
KdV means that the spectrum of L does not change in time.
Why a spectral *curve*?

Before we open an aside, we write KdV as a "reduction" of KP,

$$\frac{3}{4}u_{yy} = (u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x)_x \qquad \text{(KP)}$$

$$\partial_y \equiv 0 \ \downarrow \qquad\qquad\qquad \downarrow \partial_t \equiv 0$$

$$4u_t - u_{xxx} - 6uu_x = 0 \qquad\qquad 3u_{yy} + u_{xxx} + 6uu_x = 0$$

(KdV) (Boussinesq)

$$\text{KP} \Leftrightarrow \ D_x^2 \frac{(D_x^4 + 3D_y^2 - 4D_xD_t)F \cdot F}{2F^2} = 0$$

KP: if $\mathcal{L} = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots$

$$\partial_{t_n}\mathcal{L} = [(\mathcal{L}^n)_+, \mathcal{L}]$$

then $(u_t - 6uu_x + u_{xxx})_x = u_{yy}$
where $x = t_1, \ y = t_2, \ t = t_3$

Burchnall and Chaundy Problem

$$L \in \mathcal{D} = \{\sum_{j=0}^{n} u_j(x)\partial^j,\ u_j \text{ analytic near } x = 0\}.$$

$$\mathcal{D} \subset \mathcal{P} = \{\sum_{-\infty}^{n} u_j(x)\partial^j\},\ \text{formal pseudodifferential operators.}$$

$\mathcal{P}$ is a ring: $\qquad \partial \circ u = u\partial + u'$

$$\partial^{-1} \circ u = u\partial^{-1} - u'\partial^{-2} + u''\partial^{-3} - \dots$$

Normalize L ($\mathcal{D}$ has 2 automorphisms, change of variable and conjugation by a function)

$$L = \partial^n + u_{n-2}(x)\partial^{n-2} + \dots + u_0(x).$$

Burchnall-Chaundy problem: which L's have centralizer $\mathcal{C}_{\mathcal{D}}(L)$ which is larger than a polynomial ring $\mathbb{C}[L_1]$, $L_1 \in \mathcal{D}$?

• If ord $L > 0$ and $A, B \in \mathcal{D}$ both commute with L, then $[A, B] = 0$; in particular, $\mathcal{C}_{\mathcal{D}}(L)$ is commutative, hence every maximal-commutative subalgebra of $\mathcal{D}$ is a centralizer.

In $\mathcal{P}$ any (normalized) L has a unique nth root, $n =$ord$L$, of the form $\mathcal{L} = \partial + u_{-1}(x)\partial^{-1} + u_{-2}(x)\partial^{-2} + \dots$.

• (I. Schur, [Sc]) $\mathcal{C}_{\mathcal{D}}(L) = \{\sum_{-\infty}^{N} c_j\mathcal{L}^j, c_j \in \mathbb{C}\} \cap \mathcal{D}$.

• It follows that centralizers are 'curves': their transcendence degree over the field of coefficients is 1, and Spec $\mathcal{C}(L)$ can be regarded as an affine curve X

Direct spectral problem – Examples

The periodic case $u(x)$ $L = -\frac{d^2}{dx^2} - u(x,t)$ of period T gives Hill's equation: $Ly = \lambda y$ (G.W. Hill studied it in connection with the motion of the moon).

Well-known (elementary): the periodic ($y(x+T) = y(x)$) and antiperiodic ($y(x+T) = -y(x)$) spectra alternate:

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots$$

where λ_0 is periodic, (λ_1, λ_2) antiperiodic, etc. The intervals $(-\infty, \lambda_0]$ and $[\lambda_{2i-1}, \lambda_{2i}]$ $(i = 1, 2, \dots)$ are called gaps because the spectrum of L acting in $L^2(\mathbb{R})$ is their closed complement.

Floquet eigenfunction: $Ly = \lambda y$
$y_i(x, \lambda; x_0)$ $i = 1, 2$ normalized basis at $x = x_0$, $\quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
Monodromy Matrix $M(\lambda; x_0)$: $My = y(x+T, \lambda; x_0)$
has eigenvalues $\nu_\pm = \exp(\pm ipT)$
$p(\lambda) =$ "quasimomentum"
Floquet eigenfunction:
$T\psi_\pm = \nu_\pm \psi_\pm$ normalized by $\psi_\pm = 1$ for $x = x_0$
As $\lambda \to \infty$, asymptotically $\psi_\pm \sim \exp(\pm i\sqrt{\lambda}(x - x_0))$

Fact 1. $\psi_\pm(x, \lambda; x_0)$ is bounded when $p(\lambda)$ real, on the complement of the gaps. The values $\psi_\pm(x, \lambda; x_0)$ are values of a single-valued function meromorphic in λ, defined on the Riemann surface X which is a 2-sheeted covering of the λ line $(\lambda : 1) \in \mathbb{P}^1$, branched at the endpoints of the gaps (and

∞). We denote it by $\psi(x, P; x_0), P \in X$. The poles of ψ are independent of x and there is exactly one whose λ-coordinate belongs to each of the (finite) gaps $\gamma_j \in [\lambda_{2j-1}, \lambda_{2j}]$. The zeros of ψ depend on x.

Fact 2 (McKean-van Moerbeke). If u is finite-gap (i.e., only a finite number, g say, of pairs $(\lambda_{2i-1}, \lambda_{2i})$ are simple, while all others $\lambda_{2j-1} = \lambda_{2j}$), then the manifold of isospectral potentials is a real g-dimensional torus $S^1 \times \ldots \times S^1$. This is one connected component of $\mathrm{Jac}_{\mathbb{R}}(X)$ (real points of the Jacobi variety of the curve X), parametrized by points $P_1 + \ldots + P_g$, each $P_i = (\overline{\lambda}_1, \overline{\mu}_i), \overline{\lambda}_i$ in the i^{th} gap and each point satisfying the equation

$$\mu^2 = \prod_{s=0}^{2g} (\lambda - e_s)$$

where e_s are the simple points of the spectrum.

Moreover, Abel's coordinates on the Abelian variety $\mathrm{Jac}X$ linearize the KdV flows:

$$P_1 + \ldots + P_g \mapsto \left(\sum_1^g \int_{e_i}^P \frac{\lambda^j \, d\mu}{\sqrt{\prod_0^{2g} (\lambda - e_k)}} \right)_{0 \leq j \leq g-1} \in \mathbb{C}^g / \Lambda$$

is linear in x, t (P_i are the zeros of the Floquet eigenfunction). The spectral curve X has ring of meromorphic functions with pole only at ∞: $H^0(X \backslash \{\infty\}, \mathcal{O}_\Gamma) \cong \mathcal{C}_{\mathcal{D}}(L)$.

1. (Ince) the Lamé operator $L = -\partial^2 + a(a+1)\wp(x - x_0)$ with real, smooth potential, is finite-gap iff $a \in \mathbb{Z}$ (if a is positive the number of gaps is a).

2. (Halphen) $\partial^3 + (1-n^2)\wp(x)\partial + \frac{1-n^2}{2}\wp'(x)$ gives another finite-gap situation, where n is the number of gaps and the isospectral flow for a parameter y gives a solution to the Boussinesq equation,

$$u(x,y) = 2\frac{(1-n^2)}{3}\wp(x - cy),$$

$$u_{yy} = u_{xxx} + 6uu_x.$$

Example 1 above: equivalent to asking which Lamé operators are solutions of the BC problem.

genus 1 case: $\mathbb{C}[\wp(z), \wp'(z)] \cong \mathbb{C}[L, B]$

$$L = \partial^2 - 2\wp(z)$$

$$B = 2(\partial^3 + 3\wp(z)\partial - \frac{3}{2}\wp'(z))$$

$$B^2 = 4L^3 - g_2 L - g_3.$$

• **Definition.** The rank of a subset of $\mathcal{D}$ is the greatest common divisor of the orders of all the elements of $\mathcal{D}$.

Roughly stated, Burchnall and Chaundy's characterization of rank-1 commutative subalgebras: the correspond to affine curves, and the 'isospectral' ones corresponding to the same curve correspond to points of $\mathrm{Jac}X\backslash\Theta$. The KP deformations are linear flows on Jacobians, and they can be solved exactly in terms of theta functions,

Inverse spectral problem (Krichever)

We make the following choices:

a Riemann surface X of genus g

a point $\infty \in X$

a local parameter z^{-1} near ∞

a generic divisor $P_1 + \ldots + P_g = D$ (the condition is that $h^0(P_1 + \ldots + P_g - \infty) = 0$, no functions with a zero at ∞ and poles bounded by $P_1 + \ldots + P_g$).

Fact (Krichever). There exists a unique function $\psi(\underline{t}, P)$, the "Baker-Akhiezer (BA) function," satisfying the following conditions:

(i) near ∞, $\psi \sim \exp(\sum_{i\geq 1} t_i z^i)(1 + \sum \xi_i(\underline{t}) z^{-i})$

(ii) at finite points P of the curve, ψ has poles bounded by D and is analytic elsewhere. For such a ψ there exist unique operators K_j such that $K_j\psi = \partial_{t_j}\psi$ and these operators are a solution to the KP hierarchy, in particular $\mathcal{L}\psi = z\psi$ gives $\mathcal{L} \in \mathcal{P}$ as above.

(all statements are local in $\underline{t}$).

$\psi(\underline{t}) =$

$$e^{\left(\sum_{i\geq 1} t_i\left(\int_{P_0}^{P} \eta_i - c_i\right)\right)} \frac{\vartheta(A(P) + \sum_{i\geq 1} U_i t_i + \delta)\vartheta(A(\infty) + \delta - A(D))}{\vartheta(A(P) + \delta - A(D))\vartheta(A(\infty) + \sum_{i\geq 1} U_i t_i + \delta)}$$

δ is Riemann's constant (to make $\vartheta(A(P)+\delta-A(D))$ vanish for $P = P_j, j = 1, \ldots, g$). $U_i \in \mathbb{C}^g$ suitable vectors (to make ψ into a function of P independent of the path of integration) $c_i \in \mathbb{C}$ suitable constants (to normalize ψ as in (i) above).

Upshot:

$$u(\underline{t}) = 2\partial_x^2 \log \vartheta(\sum_{j\geq 1} t_j U_j + A(P) + \delta) + const.$$

solves the KP equation.

Remark. The $U_i = \sum_j u_{ij} \frac{\partial}{\partial z_j}$ are linear flows on $\mathrm{Jac}(X)$, so we have linearized the flows of the KP hierarchy. Geometrically, U_1 is the tangent vector to the curve $A(X)$ at $A(\infty)$, and U_j are the j^{th} hyperosculating vectors.

Introduce parameters $\underline{t} = (t_1 = x, t_2, t_3, \ldots)$, the KP hierarchy:

$$\partial_{t_j} \mathcal{L} = [(\mathcal{L}^j)_+, \mathcal{L}]$$

where $(\ )_+$ is projection $\mathcal{P} \to \mathcal{D}$, is a set of PDE's on $u_i(\underline{t})$, which turn out to be commuting Hamiltonian flows (AKS=Adler-Kostant-Symes). A solution $\mathcal{L}$ is "stationary" w.r.t. t_j iff $\mathcal{L}^j \in \mathcal{D} (\Rightarrow \partial_{t_j} \mathcal{L} = 0)$ e.g. for $j = 2$ we get KdV and for $j = 3$ we get Boussinesq, both reductions of the KP equation: $u_{yy} = (u_t + 6uu_x + u_{xxx})_x$ $(y = t_2, t = t_3)$.

More generally, let $K_j = (\mathcal{L}^j)_+$ and say that a KP solution is stationary if a nontrival combination $\sum_1^N c_j \mathcal{L}^j \in \mathcal{D}$, i.e. the corresponding time operator $\sum_1^N c_j K_j$ acts trivially.

Fact (Krichever, Novikov): An ODO $L \in \mathcal{D}$ is finite-gap iff the corresponding KP solution is stationary iff L is a solution to the Burchnall-Chaundy problem (broadly stated, skipping over technical conditions).

So, do we have all KdV/KP solutions? Not those for which infinitely many times are independent, and although some of these can be viewed as 'infinite-genus' curves, the BC ring isn't really there.

The second geometric technique is 'universal'.

Sato's infinite-dimensional Grassmann manifold

There are two ways to set it up,

1. more direct as a limit of finite-dimensional Grassmannians

2. more intrinsic, linked with the rings $\mathcal{D} \subset \mathcal{P}$

1. Let $\dim V = m + n = N$

$\mathrm{Gr}(m, V) = \{m\text{−frames in} V\}/GL(m) \hookrightarrow \mathbb{P}(\wedge^m V)$

$\xi^{(0)}, ..., \xi^{(m-1)} \mapsto \xi^{(0)} \wedge ... \wedge \xi^{(m-1)}$

If we fix a basis $e_0, ..., e_{N-1}$ of V,

$\xi^{(i)} = \xi_{0,i} e_0 + ... + \xi_{N-1,i} e_{N-1}$

$$\xi^{(0)} \wedge ... \wedge \xi^{(m-1)} = \sum_{0 \le \ell_0 < ... < \ell_{m-1} < N} \xi_{\ell_0 ... \ell_{m-1}} e_{\ell_0} \wedge ... \wedge e_{\ell_{m-1}}$$

with $\xi_{\ell_0 ... \ell_{m-1}} = \det(\xi_{\ell_i, j})_{i,j=0,...,m-1}$

FACT. A point in the ambient $\mathbb{P}(\wedge^m V)$ lies in the embedded $\mathrm{Gr}(m, V)$ $\Leftrightarrow$ its projective coordinates $\xi_{\ell_0 ... \ell_{m-1}}$ $(0 \le \ell_i < N)$ satisfy the Plücker relations:

$$\sum_{i=0}^{m} (-1)^i \xi_{k_0 ... k_{m-2} \ell_i} \xi_{\ell_0 ... \hat{\ell}_i ... \ell_m} = 0 \qquad \text{(P.R.)}$$

Therefore,

$$\mathrm{Gr}(m, V) = (\widetilde{\mathrm{Gr}}(m, V) \backslash \{0\})/GL(1)$$

where:

$\widetilde{\mathrm{Gr}}(m,V) = \{(\xi_Y)_{Y\subset\Delta_{mN}}$ satisfying the Plücker relations$\}$
is a line bundle over $\mathrm{Gr}(m,V)$,

Y is a Young diagram consisting of rows
$$\begin{array}{l} \ell_{m-1}-(m-1) \\ \vdots \\ \ell_1 - 1 \\ \ell_0 \end{array}$$

so it is contained in the rectangle Δ_{mN}.

FACT. Let $m \le m', n \le n', N' = m' + n'$

(i) If $(\xi'_Y)_{Y\subset\Delta_{m'N'}}$ satisfies (P.R.), so does its restriction to Y's within Δ_{mN}

(ii) If $(\xi_Y)_{Y\subset\Delta_{mN}}$ satisfies (P.R.), so does $(\xi'_Y)_{Y\subset\Delta_{m'N'}}$ where $\xi'_Y = 0$ unless $Y\subset\Delta_{mN}$

So, get a commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{Gr}}(m',N') & \overset{\text{project}}{\longrightarrow} & \widetilde{\mathrm{Gr}}(m,N) \\ \downarrow\text{identity} & & \downarrow\text{identity} \\ \widetilde{\mathrm{Gr}}(m',N') & \overset{\text{embed}}{\hookleftarrow} & \widetilde{\mathrm{Gr}}(m,N) \end{array}$$

Define: $\mathrm{Gr} = (\widetilde{\mathrm{Gr}}\backslash\{0\})/GL(1)$ where

$\widetilde{\mathrm{Gr}} = \{(\xi_Y)_{Y \text{ all Young diagrams}}$ satisfy all Plücker relations$\}$

$$\begin{array}{ccc} \widetilde{\mathrm{Gr}} & \overset{\text{project}}{\longrightarrow} & \widetilde{\mathrm{Gr}}(m,N) \\ \uparrow\text{dense} & & \downarrow\text{identity} \\ \widetilde{\mathrm{Gr}}^{\text{fin}} & \overset{\text{embed}}{\hookleftarrow} & \widetilde{\mathrm{Gr}}(m,N) \end{array}$$

$\widetilde{\mathrm{Gr}}^{\text{fin}} = \{(\xi)\in\widetilde{\mathrm{Gr}} : \xi_Y = 0 \text{ for almost all } Y\} = \bigcup_{m,N}\widetilde{\mathrm{Gr}}(m,N)$

$\xi_Y(t) := \sum_{\text{all } Y'} \chi_{Y'/Y}(t)\xi_{Y'}$ where
$\chi_{Y'/Y}(t) := \det(p_{\ell'_j - \ell_j}(t))$
$p_0(t) = 1,\ p_n(t) := \sum_{\nu_1+2\nu_2+3\nu_3+...=n} t_1^{\nu_1} t_2^{\nu_2} .../(\nu_1!\nu_2!...)$
Write $\chi_{Y/\emptyset}$ as χ_Y,
$\chi_Y(t) = \det(p_{\ell_i - j}(t))$ are the Schur functions

To connect with the KP hierarchy:
$w_n(x,t) := (-1)^n \frac{\xi_{\Delta_{n,1}}(x+t)}{\xi_\emptyset(x+t)}$ where $x+t = (x+t_1, t_2, ...)$
$S := 1 + w_1(x,t)\partial^{-1} + ...$
Note. The Plücker coordinate $\xi_\emptyset(t) = \sum_{\text{all } Y} \chi_Y(t)\xi_Y = \tau(\xi, t)$ is a 'generating function' for Plücker coordinates,
$\xi_Y(t) = \chi_Y(\partial_t)\xi_\emptyset(t) \qquad \partial_t := (\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, ...)$

Now by reducing to $\mathrm{Gr}(m, N)$ and checking, every $\xi_Y(t)$ satisfies Plücker relations, so have a dynamical system on $\widetilde{\mathrm{Gr}}$, which satisfies the KP hierarchy: if

$$\mathcal{L} = S\partial S^{-1}$$

then $\partial_{t_n} S = B_n S - S\partial^n$, where $B_n := (S\partial^n S^{-1})_+$,
$\iff [\partial_{t_n} - B_n, \partial_{t_k} - B_k] = 0 \iff \partial_{t_n}\mathcal{L} = [(\mathcal{L}^n)_+, \mathcal{L}]$

CONCLUSION (Sato)

Although any $f(t) \in \mathbb{C}[[t_1, t_2, ...]]$ admits a formal expression of the form $\sum_Y c_Y \chi_Y(t)$, where the coefficients are

$$c_Y = \chi_Y(\partial_t) f(t)|_{t=0},$$

it represents the τ function for some $\xi \in \widetilde{\mathrm{Gr}} \Longleftrightarrow$ its coefficients satisfy the Plücker relations,

$$\sum_{i=0}^{m}(-1)^i \chi_{k_0\ldots k_{m-1}\ell_i}\left(\frac{\partial_t}{2}\right)\chi_{\ell_o\ldots\hat{\ell}_i\ldots\ell_m}\left(-\frac{\partial_t}{2}\right)\tau\cdot\tau = 0 \tag{P.R.}$$

which is the KP hierarchy in Hirota bilinear fom.

2. Other Model

Let $V := \mathcal{P}/\mathcal{P}x \cong \mathcal{P}_{\text{const}} = \{\sum_{-\infty<i<<\infty} a_i\partial^i \quad a_i \in \mathbb{C}\}$

equipped with the induced filtration $V^{(i)}$ (by order) and define

Gr= { vector subspaces W of V s.t. $\dim(W\cap V^{(0)}) = \dim V/(W+V^{(0)}) < \infty\}$,

'same size' as the reference subspace

$\{\sum_{\nu\le 0} c_\nu e_\nu : \ c_\nu \in \mathbb{C}\} = V^{(0)}$

generic: $\mathrm{Gr}^{\emptyset} \underset{\text{open dense}}{\subset} \mathrm{Gr} \Longleftrightarrow V = W \oplus V^{(0)} \Longleftrightarrow \xi_{\emptyset} \neq 0$

In standard basis of V, $e_i := \partial^{-i-1} \bmod \mathcal{P}x$, $i \in \mathbb{Z}$

action: $\begin{array}{c} xe_i = (i+1)e_{i+1} \\ \partial e_i = e_{i-1} \end{array}$ gives V a $\mathcal{P}$-module structure

Let Λ be the shift operator: $\partial e_i = e_{i-1}$, then

$$\xi(t) = e^{t_1\Lambda + t_2\Lambda^2 + \ldots}\xi$$

so, this 'linearizes' the flows!

Link with differential operators:

There is a 1:1 correspondence between elements of $\mathrm{Gr}^{\emptyset}$ and $\mathcal{D}$-submodules $\mathcal{I}$ of $\mathcal{P}$ with the property:

$$\mathcal{P} = \mathcal{I} \oplus \mathcal{P}^{(-1)}$$

($\Rightarrow$ cyclic, i.e. $\mathcal{I} = \mathcal{D}S$)

$$\mathcal{I} \mapsto W = S^{-1}V^{(0)} = \{v \in V : \ \mathcal{I}v \subset V^{(0)}\}$$

$$W \mapsto \mathcal{I} = \{A \in \mathcal{P} : \ AW \subset V^{(0)}\}$$

so KP viewed as deformation of $\mathcal{D}$ modules

Under $\partial^{-1} \leftrightarrow z$,
$\mathcal{P}_{\mathrm{const}} \cong \mathbb{C}((z)) = \mathbb{C}[[z]][z^{-1}]$ (formal Laurent series)
consider $\mathcal{A} \subset \mathcal{P}_{\mathrm{const}}$ such that $\mathcal{A} \cap \mathbb{C}[[z]] = \mathbb{C}$
Then $\mathrm{Gr}^{\mathcal{A}} = \{W \in \mathrm{Gr} \text{ s.t. } f(\Lambda)W \subset W \ \forall f(z) \in \mathcal{A}\}$
is stable under the KP flows

Note. $\mathcal{A} = \mathbb{C}[z^{-1}] \Rightarrow$ KdV
when $\mathcal{A}$ 'large' i.e. $\dim \mathbb{C}((z))/(\mathcal{A} + \mathbb{C}[[z]])$ finite,
ex. $\mathcal{A} = \mathbb{C}[\wp(z), \wp'(z)]$, then $\mathcal{A} = \mathcal{O}(X \backslash P_{\infty})$

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PART II

Differential equations integrable by algebraic geometry: Two open directions and their interpretation in differential algebra

We now have 'all' KP solutions, including non-convergent [e.g. Kac-Schwarz] and non-algebro-geometric [e.g. De Concini-Johnson]. Which came from algebraic curves?

Review what we did:

the KP hierarchy is a sequence of deformations for $\mathcal{L} = S\partial S^{-1}$, $S \in \mathcal{P}$, the Baker (eigen)function $\psi = Se^{xz}$ turns differentiation into multiplication.

Corresponding data in the Grassmannian,
$W = \mathcal{D}S$, $\in \mathrm{Gr}V$, $V = \mathcal{P}/\mathcal{P}x$ and these $W \oplus \mathcal{P}^{(-1)} = \mathcal{P}$.

$$\partial^{-1} \leftrightarrow z \quad \mathcal{P}^{(-1)} \leftrightarrow H_+$$

Analytic version [SW]: $H = \mathrm{L}^2(S^1, \mathbb{C})$,
$W \in \mathrm{Gr}$ if $\pi_+ : W \to H_+$ is Fredholm, of index 0 in $\mathrm{Gr}^\emptyset$,
$\pi_- : W \to H_-$ is compact.

Krichever map:

$(X, P_x, z, \mathcal{E}, \phi)$, with ϕ a local trivialization on D_∞ of the line bundle $\mathcal{E}$, such that $h^0(\mathcal{L}) = 1, h^0(\mathcal{E} \otimes \mathcal{O}(-P_\infty)) = 0$, the space W is defined as the Hilbert closure of the space of holomorphic sections of $\mathcal{E}$ over $X \backslash D_\infty$ (viewed as elements of $L^2(S^1, \mathbf{C})$ via the trivialization over $|z| = 1$). The 'abelian' KP flows coincide with the 'Grassmannian' KP flows, $W \mapsto e^{\sum_1^\infty t_i z^i} W$, where the Baker function ψ_W is defined by $g^{-1}\psi(\underline{t}, z) = \pi_+^{-1}(1)$ via $\pi_+ \colon g^{-1}W \to H_+$; then $\psi(\underline{t}, z) = e^{\Sigma t_j z^j}(1 + O(z^{-1}))$ belongs to W for all time, and if $\psi = Se^{xz}$ then $\mathcal{L} = S^{-1}\partial S$ is a solution of the KP hierarchy, where $g = e^{\Sigma t_j z^j} \in \Gamma_+$, $|t|$ small, so that $g^{-1}W$ is still transverse (its projection to H_+ is an isomorphism) if W is transverse.

Given a $W \in GrH^{(1)}$, we associate to it the ring A_W of analytic functions $a(z)$ which have a finite number of positive Fourier coefficients and are such that $a(z)W^{alg} \subset W^{alg}$; then W comes from the Krichever map precisely if A_W contains functions of any sufficiently high order; the BC ring is given by $\{a(\mathcal{L})\}$, where $\mathcal{L}$ is the KP solution constructed via the Grassmannian and evaluated at $t_j = 0$ for $j > 1$.

So, where are the solutions that Grünbaum found (non-vanishing as $x \to \infty$)?

$$2u = -\frac{4(3t+2)^2}{[(3t+2)x - 2y^2]^2} - \frac{4x}{3t+2}$$

In order to address the higher-rank/dimension problem, let's revisit the BC theory.

Recall for the Lamé operators:

If $c \in \mathbf{C}$ is a constant, $L = \partial^2 - c\wp(x)$ is a BC solution iff $c = n(n+1)$ with n an integer greater than zero – if this is the case, the centralizer $\mathcal{C}_{\mathcal{D}}(L)$ is the affine ring of a hyperelliptic curve of genus n, given by an equation: $\mu^2 = \lambda^{2n+1}+$ lower order.

[TV] gives the following generalization: if $\omega_i (0 \le i \le 3)$ are the half periods of $\wp, c_i \in \mathbf{C}$ are constants, $u = -2\sum_{i=0}^{3} c_i \wp(x - \omega_i)$, and $L = \partial^2 + u$, then L is a BC solution iff each c_i is a triangular number $a_i(a_i + 1)/2$ for a_i some positive integers.

However, there are no BC solutions of order 2 with polynomial coefficients. Dixmier constructs a genus-1 maximal commutative subring of the Weyl algebra in two generators $\mathbf{C}[p, q]$ with multiplication rule defined by the commutator $[p, q] = 1$, which can be viewed as a subring of $\mathcal{D}$, by letting $p = \partial$ and $q = x$.

Define $u = p^3 + q^2 + \alpha,\ v = \frac{1}{2}p,\ L = u^2 + 4v,\ B = u^3 + 3(uv - vu)$; then $\mathcal{C}(L) = \mathbf{C}[L, B]$ and $B^2 - L^3 = -\alpha$, as shown in [D]. By the assignment $p = \partial,\ q = x$ we obtain $L, B \in \mathcal{D}$ of order 6,9, but the automorphism $\partial \mapsto -x,\ x \mapsto \partial$ will turn the orders into 4,6. Moreover, it will still be true that $\mathcal{C}_{\mathcal{D}}(L) = \mathbf{C}[L, B]$, the affine ring of the curve $\mu^2 = \lambda^3 - \alpha$; in particular, L is a BC solution.

These are rings of rank 3, 2, resp. To understand the role of the rank, let's go back to the BC construction.

Lemma [BC1]. *If* $[L,B]=0$ *then there exists a polynomial in two variables* $f(\lambda,\mu)\in \mathbf{C}[\lambda,\mu]$ *such that* $f(L,B)\equiv 0$, *if we assign "weight"* $na+mb$ *to a monomial* $\lambda^a\mu^b$ *where* $n=$ *ord* L, $m=$ *ord* B, *then the terms of highest weight in* f *are* $\alpha\lambda^m+\beta\mu^n$ *for some constants* α,β.

Proof and Construction: The idea is that by commutativity B acts on V_λ, the n-dimensional vector space of solutions $y(x)$ of $Ly=\lambda y$ (L is regular); $f(\lambda,\mu)$ is the characteristic polynomial of this operator; to see that $f(L,B)\equiv 0$ it is enough to remark that $f(\lambda,\mu)=0$ iff L,B have a "common eigenfunction": $\begin{cases} Ly=\lambda y \\ By=\mu y \end{cases}$ hence $f(L,B)$ would have an infinite-dimensional kernel (eigenfunctions belonging to distinct eigenvalues $\lambda_1,\ldots,\lambda_k$ are independent by a Vandermonde argument). But what brings out the algebraic structure of the problem, and of the polynomial f, is the construction of the "BC matrix" M: if

$$L-\lambda = u_{0,0}+u_{0,1}\partial+\ldots+\partial^n$$
$$(0=u_{0,n+1}=u_{0,n+2}=\ldots)$$
$$\partial\circ(L-\lambda)=u_{1,0}+u_{1,1}\partial+\ldots$$
$$\ldots$$
$$\partial^{m-1}\circ(L-\lambda)=u_{m-1,0}+\ldots$$
$$B-\mu=u_{m,0}+u_{m,1}\partial+\ldots+\partial^m$$
$$\ldots$$
$$\partial^{n-1}\circ(B-\mu)=u_{m+n-1,0}+\ldots$$

then $M = [m_{ij}]$ with $m_{ij} = u_{i-1,j-1}$ $(i = 1, \ldots, m+n;\ j = 1, \ldots, m+n)$ is such that det $M = f(\lambda, \mu)^r$, where $r = \gcd(\mathrm{ord} L, \mathrm{ord} B)$.

Example:

$$L = \partial^2 - \frac{2}{x^2}$$
$$B = \partial^3 - \frac{3}{x^2}\partial + \frac{3}{x^3}$$

$$\det \begin{bmatrix} -\lambda - \frac{2}{x^2} & 0 & 1 & 0 & 0 \\ \frac{4}{x^3} & -\lambda - \frac{2}{x^2} & 0 & 1 & 0 \\ -\frac{12}{x^4} & \frac{8}{x^3} & -\lambda - \frac{2}{x^2} & 0 & 1 \\ -\mu + \frac{3}{x^3} & -\frac{3}{x^2} & 0 & 1 & 0 \\ -\frac{9}{x^4} & -\mu + \frac{9}{x^3} & -\frac{3}{x^2} & 0 & 1 \end{bmatrix} = \mu^2 - \lambda^3.$$

For the higher-rank Grassmannian construction, we associate to a rank r BC algebra $\mathcal{A}$ a curve X with a smooth point P_∞, a local parameter z^{-1}, a rank r vector bundle with a local trivialization near P_∞ given by completing over P_∞ the basis $s_j \colon y \mapsto y^{(j)}(0)$ $j = 0, \ldots, r-1$ valid for finite P near P_∞. This will give us a $W \in GrH^{(r)}$ by the usual recipe (completion of sections on $X \backslash D_\infty$).

We produce a vector Grassmannian from Gr by 'interleaving' the Fourier coefficients:

$$[f_0(z), \ldots, f_{r-1}(z)] \mapsto f(z) = \sum_{i=0}^{r} z^i f_i(z^r),$$

with inverse:

$$f(z) \mapsto f_i(z) = \frac{1}{r} \sum_{\zeta^r = z} \zeta^{-i} f(\zeta)$$

Notice that the isometry we just described preserves the gradation by making the standard basis z^j correspond to the basis $\{\underline{e}_i\}_{i \in \mathbf{Z}}$ where if $i = ar + b$ with $a \in \mathbf{Z}$, $0 \leq b \leq r-1$, $\underline{e}_i$ is the vector with b-th entry z^a and all other entries zero; thus the vector corresponding to $\psi(0, z)$ is $\underline{e}_0 + \sum_{i=1}^{\infty} a_i(0) \underline{e}_{-i}$. The matrix

$$\zeta = \begin{bmatrix} 0 & 1 & 0 & \ldots & \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & & 1 \\ z & 0 & \ldots & & 0 \end{bmatrix}$$

gives the shift $\underline{e}_i = \underline{e}_0 \zeta^i$.

The subspace W corresponding to the quintuple $(X, P_\infty, z, \mathcal{E}, \phi)$ belongs to the big cell $\mathrm{Gr}^{\emptyset}$ if and only if (i) $h^0(\mathcal{E}) = r, h^1(\mathcal{L}) = 0$, (ii) the values of the global (holomorphic) sections span the fibre of $\mathcal{E}$ over P_∞.

However, the point $W \in GrH^{(r)}$ alone does not give sufficiently many interesting KP solutions; the generalization of the rank 1 construction is the following: choose an arbitrary r-th order operator of the form $L_0 = \partial^r + u_{r-2}(x)\partial^{r-2} + \ldots + u_0(x)$ and define its normalized (at $x = 0$) Wronskian matrix $\Psi_0(x, z)$, so that the first row $\underline{\psi}_0 = (\psi_0, \ldots, \psi_{r-1})$ is a basis of solutions of $L_0 \psi = z\psi$, and the jth row is given by $\underline{\psi}_0^{(j-1)}$.

For this, $\frac{\partial \Psi_0}{\partial x} = (\zeta - U)\Psi_0$ where $U = \begin{bmatrix} 0 & \ldots & & 0 \\ \ldots & & & \\ 0 & \ldots & & 0 \\ u_0 & \ldots & u_{r-2} & 0 \end{bmatrix}$.

Notice that if the coefficients of L_0 also depend on a sequence of time parameters $\underline{t}$ in such a way that $L_0^{1/r}$ is a solution of the KP hierarchy, then we obtain a matrix $\Psi_0(\underline{t}, z)$ for the eigenvalue problem

$$\begin{cases} L_0 \psi = z\psi \\ \\ \partial_{t_j} \psi = (L_0^{j/r})_+ \psi. \end{cases}$$

Now we introduce the the $\mathcal{D}$-module M: let $M(\Psi_0)$ be the space of formal expressions $(\sum_{i=-\infty}^{N} b_i(x)\underline{e}_i)\Psi_0(x, z)$; it is easy to see by using the gradation that the (obvious, left-)action of ∂ on $M(\Psi_0)$ is invertible and that $M(\Psi_0)$ is a free Ψ-module of rank 1 – as a generator, we can take any element with invertible leading coefficient b_N, for instance, $\underline{\psi}_0 = \underline{e}_0 \Psi_0$.

Theorem [P-Wilson]. *(i) If we define (for $|\underline{t}|$ small enough) the Baker vector $\underline{\xi}(\underline{t}, z)$ of (W, L_0) by $\pi_+^{-1}|_{W\Psi_0^{-1}}(\underline{e}_0)$ i.e. by the conditions that $\underline{\psi} = \underline{\xi}\Psi_0$ has the form $(\underline{e}_0 + \sum_{i=1}^{\infty} a_i(\underline{t})\underline{e}_i)\Psi_0(\underline{t}, z)$ and belongs to W for each fixed time, and if S is the unique element of $\mathcal{P}$ such that $\underline{\psi} = S\underline{\psi}_0$, then $L = SL_0S^{-1} \in \Psi$ is such that $L\underline{\psi} = z\underline{\psi}$ and $L^{1/r}$ is a solution of the KP hierarchy.*

(ii) $W \in GrH^{(r)}$ comes from geometric data exactly when its ring A_W contains a function of any sufficiently large order (notice that the corresponding $A_{\tilde{W}}$ of the interleaved image $\tilde{W} \in GrH^{(1)}$ has then rank r since $z \mapsto z^r$); in this case, by setting all higher times $= 0$, we obtain a rank r commutative subalgebra of $\mathcal{D} = \{a(L), a(z) \in A_W\}$ (where L is defined in (i)), which is isomorphic to $H^0(X\backslash\{P_\infty\}, \mathcal{O}_X)$.

Problems with this:

(1) Explicit solutions

(2) Fake rank

(3) KP hierarchy

Partial answer only for $g = 1$ (rank 2, 3).

Rank 2.

Geometry: $g = 1 = \text{genus}\ (X)$, $r = 2$ explicit formulas (Krichever-Novikov)

$$X\colon\ \mu^2 = \lambda^3 + \frac{g_2}{4}\lambda + \frac{g_3}{4} \cup \{\infty\}$$

$$L = (\partial^2 + \frac{1}{2}c_2)^2 + c_1\partial + \partial c_1 + c_0, \quad B = (L^{3/2})_+$$

$$c_0 = -\wp(\gamma_1) - \wp(\gamma_2), \quad \gamma_i = \gamma_0 - (-1)^i c(x)$$

$$c_1 = c'(\wp(\gamma_2) - \wp(\gamma_1))$$

$$c_2 = \left(\frac{c''}{c'} - 2c'\phi\right)' - \frac{1}{2}\left(\frac{c''}{c'} - 2c'\phi\right)^2 - \frac{1}{2c'^2}(= 2u)$$

$$\phi = \zeta(\gamma_1) - \zeta(\gamma_2) - \zeta(2c)$$

Algebra (Grünbaum)

$$\frac{\partial u}{\partial y} = 2\frac{\partial^2 g}{\partial x^2}$$

$$g = \frac{\partial}{\partial y}\ln[\wp(y) - \wp(c(x,t))]$$

Krichever-Novikov equation:

$$c_t = \frac{1}{4}c_{xxx} + \frac{3}{8}\frac{1-c_x^2}{c_x} - \frac{3}{2}\wp(2c){c_x}^3 \qquad (\mathrm{KN}_2)$$

$v = \wp(c) \Rightarrow$

$$\frac{v_t}{v_x} = \frac{1}{4}\{v,x\} + \frac{3}{2}\frac{v^3 + \frac{1}{4}g_2 v + \frac{1}{4}g_3}{v_x^2}$$

$$\{v,x\} = v_{xxx}v_x^{-1} - \frac{3}{2}v_{xx}^2 v_x^{-2}$$

Questions:

1. Dictionary between c and L_0.
2. Hierarchy of evolutions for c.

Both answered by Darboux (transference) in the singular case ($\lambda^3 + \frac{g_2}{4}\lambda + \frac{g_3}{4}$ has repeated roots).

Two Darboux′s

Proposition 1 [Latham-P.] $G = \gcd(L - \lambda, B - \mu)$, $L - \lambda = QG, \tilde{L} - \lambda = GQ$, then

$$\tilde{\gamma}_i = \gamma_0 + \rho - (-1)^i c(x) \quad \text{when} \quad \lambda = \wp(\rho), \mu = \frac{1}{2}\wp'(\rho).$$

In particular, L can be made formally self-adjoint ($c_1 = 0$).

Proposition 2. (Sokolov) The KN equation is equivalent to

$$\partial_t L = [(L^{3/4})_+, L]$$

$$\text{when} \quad L = (\partial^2 + \frac{1}{2}\phi)^2 + c_0, \quad c_0 = -2\wp(c) = -2v,$$

$$\phi = -2\partial_x^2 \ln[\partial_x \wp(c)] + \{\wp(c); x\} - 1/(2c_x^2)$$

Comment: the self-adjoint case of commuting operators of orders 4 and 6 defines the Lax operator for the KN equation. Thus, a solution of the KP equation (KN, resp.) determines a solution of the KN equation (KP, resp.) by transference at $\mp P_0 = (\wp(y), \mp\frac{1}{2}\wp'(y))$.

Proposition 3 [Latham-P.] If X is singular,

$$\mu^2 = (\lambda - e_1)^2(\lambda + 2e_1) = \frac{h(\lambda)}{4},$$

then for a self-adjoint L transference at $(e_1, 0)$ gives

$$\tilde{L} = (\partial^2 + \frac{1}{2}V)^2 - 2e_1,$$

where

$$V = 4\partial_x^2 \ln[v - e_i] - 2\partial_x^2 \ln[v_x] + \{v; x\} - \frac{1}{2}\frac{h(v)}{v_x^2}$$

solves the KdV equation

$$V_t - \frac{3}{4}VV_x - \frac{1}{4}V_{xxx} = \frac{3}{2}\frac{h'(e_1)v_x}{2(v - e_i)^2} = 0$$

(as predicted by Svinolupov-Sokolov-Yamilov: singular KN $\leftrightarrow$ KdV).

Conclusion: We have a recipe (L_0, curve, bundle) $\leftrightarrow$ KP = (curve, Tyurin parameters: γ_0, $c(x, \underline{t})$) by taking $L_0^2 = \tilde{L}, c = \sqrt{v^{-1}}$.

Remark. This was Burchnall-Chaundy's original idea, start with $\mathbb{C}[\partial]$, perform a Darboux transformation by an eigenfunction of ∂^2 which is not an eigenfunction of $\partial, \psi = x$, then

$$\mathcal{A} = \left[\left(\partial - \frac{\psi'}{\psi} \right) \mathbb{C}[\partial] \left(\partial - \frac{\psi'}{\psi} \right)^{-1} \cap \mathcal{D} \right] \cong \mathbb{C}[x^2, x^3]$$

is a "nonreducible" algebra produced from a reducible one. This gives true higher-rank examples!

$$L_0 = \partial^2 - \frac{2a}{x^2}, \quad a = \frac{1}{2}\frac{N}{2}\left(\frac{N}{2}+1\right), \; N \text{ odd } = \frac{2nm-1}{2}$$

("half-triangular" numbers)

$$\begin{aligned} L &= x^{-2n}\delta(\delta - 2m)(\delta - 4m)\ldots(\delta - 2m(2n-1)) \\ B &= x^{-2m}\delta(\delta - 2n)(\delta - 4n)\ldots(\delta - 2n(2m-1)) \\ \delta &= x\partial \end{aligned}$$

Application
Grünbaum's KP solution

$$2u = -\frac{4(3t+2)^2}{[(3t+2)x - 2y^2]^2} - \frac{4x}{3t+2} (= c_2)$$

has "true" rank 2 (it is the Darboux transform of an Airy operator $L_0 = \partial^2 - \dfrac{2x}{3t+2}$)

Higher-dimensional spectral varieties/
Commutative Algebras of partial differential operators

Geometry:
(Sato-Nakayashiki)

$$\partial_i = \frac{\partial}{\partial x_i} \quad 0 \le i \le r, \quad |\alpha| = \alpha_0 + \ldots + \alpha_{r-1}$$

$$\mathcal{D} = \left\{ \sum_{|\alpha|<<\infty} a_\alpha(x)\partial^\alpha, \; a_\alpha(x) \in \mathbf{C}[[x]], \; \alpha \in \mathbf{N}^r \right\}$$

micro-differential operators: codirection x_0

$$\mathcal{P} := \left\{ \sum_{|\alpha|<<\infty} a_\alpha(x)\partial^\alpha, \quad \alpha \in \mathbf{Z} \times \mathbf{N}^{r-1} \right\}$$

$\Theta \subset A$, g dimensional Abelian variety
$\nabla_j = \frac{\partial}{\partial x_j} - \int \eta_j \qquad \eta_j$ 2nd kind

$\mathcal{L}$ line bundle over A,

$$\phi_{\mathcal{L}} : \mathcal{A}_\Theta \to M(g! \times g!, \mathcal{D})$$

No KP deformations
More recent work: Mironov, Parshin, Osipov, Rothstein

PDO case
Commutative rings of PDOs

$$\mathcal{D} = \mathbf{C}(x_1, \ldots, x_n)[\partial_1, \ldots, \partial_n]$$

$$\mathcal{D}_0 = \mathbf{C}[\partial_1, \ldots, \partial_n] \subset \mathcal{D}$$

Construction (Kasman-P.):
A maximal-commutative ring $R \subset \mathcal{D}$ which is not finitely generated.

Step I.
Suppose $p(\partial_1, \ldots, \partial_n) \in \mathcal{D}_0$ factors $= L \circ K, \quad L, K \in \mathcal{D}$
let $R(K) := (K \circ \mathcal{D}_0 \circ K^{-1}) \cap \mathcal{D}$

Th. $R(K)$ is maximal commutative.
(use $e^{x_1 z_1 + \ldots + x_n z_n}$)

Step II. Let R_λ = polynomials in $\mathbf{C}[x, y]$ s.t. q_x, q_y, q_{xy} are divisible by $(xy - \lambda)$
$(= \mathbf{C}[\omega_i(xy-\lambda)^3],\ \omega_i$ a basis of $\mathbf{C}[x, y])$

$$p = (\partial_1\partial_2 - \lambda)^3, \quad K = x_1x_2(\partial_1\partial_2 - \lambda) \circ \frac{1}{x_1x_2}$$

$$L = \partial_1^2\partial_2^2 + \frac{1}{x_1}\partial_1\partial_2^2 - x_1^{-2}\partial_2^2 + \frac{1}{x_2}\partial_1^2\partial_2$$
$$+\frac{1 - 2\lambda x_1x_2}{x_1x_2}\partial_1\partial_2 + \frac{-1 - \lambda x_1x_2}{x_1^2x_2}\partial_2$$
$$-x_2^{-2}\partial_1^2 + \frac{-1 - \lambda x_1x_2}{x_1x_2^2}\partial_1 + \lambda^2 + \frac{1}{x_1^2x_2^2} + \frac{\lambda}{x_1x_2}$$

Th. $R(K) \simeq R_\lambda$

Construction: differential resultant
(F.S. Macaulay for polynomials)

$L_1, \ldots, L_{n+1}$ orders $\ell_1, \ldots, \ell_{n+1}$

$$N = 1 + \sum_i (\ell_i - 1)$$

R: each row $\partial^\alpha \circ (L_i - \mu_i)$ $1 \leq |\alpha| \leq \binom{n+N-l_i}{n}$

Resultant=gcd of all maximal minors = 0 if $L_1, \ldots, L_{n+1}$ commute.

Conjecture: indep. of $x_1, \ldots, x_n$.

More recent work: Quantum integrable systems: Chalykh, Veselov, Braverman-Etingof-Gaitsgory and Chalykh-Etingof-Oblomkov

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