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The converse of the Ritt-Raudenbush theorem for commutative differential Hopf algebras

For the Galois theory of parametrized differential equations and linear differential algebraic groups, see [1], which is joint work with Michael Singer.The formulation of the theory of parametrized Picard-Vessiot extensions in [1] requires the re-casting of the theory of linear differential algebraic groups in the language of Hopf algebras. This is joint work with Jerry Kovacic.

1 Some basic notions.

Let k be a differential field of characteristic zero, with family $\partial = (\partial_1, \ldots, \partial_m)$ of commuting derivation operators.

Notation 1.1. The operators in the commutative monoid Θ generated by ∂ are denoted, using multi-index notation, by $\partial^e := \partial_1^{e_1} \cdots \partial_m^{e_m}$, $e_i \in \mathbb{Z}_{\geq 0}, i = 1, \ldots, m$.

The prefix ∂- indicates the differential structure. If R is a ∂-ring, R^∂ is the subring of constants of R.

Convention: We will assume throughout that all ∂-rings R are commutative.

In the following, we emphasize that if a k-algebra is equipped with a differential structure, *finitely generated* means *differentially finitely generated.*

- A ∂-ideal I of a ∂-ring R is *finitely generated* if there exists a family $(\alpha_1, \ldots, \alpha_r)$ of elements of I such that I is the ideal generated by the family $\partial^e \alpha = (\partial^e \alpha_j)_{e \in \mathbb{Z}^m_{\geq 0}, j=1,\ldots,r}$ of all derivatives of $\alpha_1, \ldots, \alpha_r$. We denote I by $[\alpha_1, \ldots, \alpha_r]$.

- A *radical ∂-ideal* I of a ∂-ring R is *finitely generated* if there is a finite family $(\alpha_1, \ldots, \alpha_r)$ of elements of I such that $I = \sqrt{[\alpha_1, \ldots, \alpha_r]}$. Since the characteristic of R is zero, $\sqrt{[\alpha_1, \ldots, \alpha_r]}$ is always a ∂-ideal.

- Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a family of elements of a ∂-k-algebra. The ∂-k-algebra *generated by* α is the k-algebra $k[\partial^e \alpha]$ generated by the family of all derivatives of the elements of α. We denote it by $k\{\alpha\}$. The ∂-k-algebra R is said to be *finitely generated.* It may, of course, be infinitely generated as a k-algebra.

- The ∂-polynomial ring is denoted by $k\{y_1, \ldots, y_n\}$, or, $k\{y\}$, where $y = (y_1, ..., y_n)$ is a family of ∂-indeterminates.

Remark 1.2. Let $m = n = 1$. In 1934 [5], Ritt gave an example of a ∂-ideal I in $\mathbb{Q}\{y\}$ that is not finitely generated. $I = [\partial y \partial^2 y, \partial^2 y \partial^3 y, ..., \partial^e y \partial^{e+1} y, \ldots]$.

Theorem 1.3. *The Ritt-Raudenbush Basis Theorem (1934) If R is a finitely generated ∂-k-algebra, every radical ∂-ideal I is finitely generated, i.e., there exists a family $(\alpha_1, \ldots, \alpha_r)$ of elements of I such that $I = \sqrt{[\alpha_1, \ldots, \alpha_r]}$.*

Remark 1.4. It is not hard to show that $(\partial^{e+1} y \partial^{e+2} y)^2 \in [\partial^e y \partial^{e+1} y]$, $e = 1, 2, \ldots$. Thus, $I = \sqrt{[\partial y \partial^2 y]}$.

Definition 1.5. A ∂-k-algebra R is *∂-Noetherian* if every radical ∂-ideal of R is finitely generated.

The Ritt-Raudenbush Basis Theorem $\Longrightarrow$ a commutative finitely generated ∂-k-algebra R is ∂-Noetherian. We shall sketch the proof of the converse of the Basis Theorem for *certain* commutative ∂-k-algebras R, called *Hopf algebras*.

2 ∂-k-vector spaces

Definition 2.1. A *∂-k-vector space* is a k-vector space V, together with a homomorphism $\Theta \to Lin_{k^\partial}(V, V)$ such that for $\alpha \in k$, $v \in V$, $i = 1, \ldots, m$, $\partial_i(\alpha v) = \alpha \partial_i v + \partial_i \alpha\, v$. ∂-k-vector subspaces are defined in the obvious way.

Example 2.2.

1. Let $V = k\{y_1, \ldots, y_n\}_1 = \left\{ L = \sum_{i=1}^n \sum_{e \in \mathbb{Z}^m_{\geq 0}} \alpha_{ie} \partial^e y_i : \alpha_{ie} \in k \right\}$.

2. Let $V = sl(n, K) = \{a = [a_{ij}] \in GL(n, K) : Tr(a) = 0\}$.
   $\partial_l a = [\partial_l a_{ij}]$. $\sum \partial_l a_{ii} = \partial\left(\sum a_{ii}\right) = 0$.
   Set $n = 2$. $e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ form a basis of $sl(2, K)$.
   Let $a = a_{12}e_{12} + a_{21}e_{21} + a_{11}h$. Then, $\partial a = (\partial a_{12})e_{12} + (\partial a_{21})e_{21} + \partial(a_{11})h$.

Definition 2.3. A k-linear map φ from a ∂-k-vector space V to a ∂-k-vector space V' is *∂-k-linear* if $\varphi \circ \partial_i = \partial_i \circ \varphi, i = 1, \ldots, m$.

The kernel of φ is a ∂-k-subspace of V and the image of φ is a ∂-k-subspace of V'.

Fix a ∂-k-vector space V.

Let $X \subset V$. The ∂-k-subspace of V generated by X is the intersection of the family of ∂-k-subspaces of V containing X, and, is denoted by $\langle X \rangle$. $\langle X \rangle = \left\{ \sum_{x \in X, e \in \mathbb{Z}^m_{\geq o}} \alpha_e \partial^e x : \alpha_e \in k \right\}$.

Lemma 2.4. *Let W be the k-subspace of V generated by $X \subset V$. Then, $\langle X \rangle = \langle W \rangle$.*

Corollary 2.5. *If W is a finite dimensional k-subspace of V, then, $\langle W \rangle$ is a finitely generated ∂-k-subspace of V. Let $X = \{x_1, \ldots, x_r\}$ be a finite basis for W. Then, X is a finite set of generators of $\langle W \rangle$.*

Remark 2.6. $\langle W \rangle = \left\{ \sum_{i=1}^{r} (\sum_{e \in \mathbb{Z}^m_{\geq o}} \alpha_{ei} \partial^e) x_i : a_e \in k \right\} = k[\partial] x_1 + \cdots + k[\partial] x_r$, where $k[\partial]$ is the non-commutative ring of linear differential operators in ∂, with coefficients in k.

Remark 2.7. If V and V' are ∂-k-spaces, then, $V \otimes_k V'$ is a ∂-k-space in a natural way: $\partial_i (v \otimes v') = \partial_i v \otimes v' + v \otimes \partial_i v', i = 1, \ldots, m$. We define the *twist map* $\tau : V \otimes V \to V \otimes V$ to be the ∂-k-linear endomorphism defined by $\alpha \otimes \beta \longmapsto \beta \otimes \alpha$.

3 Linear differential algebraic groups

Let K be a ∂-closed differential extension field of k.

Definition 3.1. A subgroup G of $GL(n, K)$ is a *∂-k-subgroup* if its underlying set is the set of zeros in $GL(n, K)$ of a system of differential polynomials in the entries of the matrices. Call G a *linear ∂-k-group.*

The Ritt-Raudenbush basis theorem says that a finite subsystem suffices to define G.

Example 3.2.

1. The multiplicative group $GL(1, K) = \mathbb{G}_m$ and additive group $\mathbb{G}_a$ of the field K. We represent $\mathbb{G}_a$ as the matrix group $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $a \in K$. The general linear group $GL(n)$ of $n \times n$ matrices with entries in K.

2. K an ordinary differential field with derivation operator ∂. The subgroup G of $\mathbb{G}_m$ defined by the equation $\left(y\partial^2 y - (\partial y)^2\right) = 0$.

3. K an ordinary differential field with derivation operator ∂. $G = K \times K$, with the multiplication law

$$(u_1, u_2) \cdot (v_1, v_2) = (u_1 + v_1, u_2 + v_2 + \sum_{i<j} \alpha_{ij} \partial^i u_1 \partial^j v_1), \alpha_{ij} \in k.$$

Let G be a ∂-K-subgroup of $GL(n, K)$.

Let $y = [y_{ij}]_{ij}$ be a matrix of differential indeterminates.

Let $f_1 = 0, \ldots, f_r = 0,$ be the defining differential polynomials of G.

Let $I = \sqrt{[f_1, \ldots, f_r]}$ in $K\{y\}$, and, $K\{\overline{y}\}$ the residue class ring $K\{y\}/I$.

$R = K\{\overline{y}, \overline{y}^{-1}\}$ is a finitely generated ∂-K-algebra.

In addition, R is equipped with another structure–*∂-K-Hopf algebra*– which we will explore today.

Call R the *Hopf algebra representing* G.

Let $K \subset E \subset L$, L a finitely generated ∂-K-field ,E a ∂-subfield of L.

E is finitely generated as a ∂-K-field.(Kolchin [2]).

Let $K \subset S \subset R$, R a finitely generated ∂-K-algebra, S a ∂-subalgebra of R.

S need not be a finitely generated ∂-K-algebra.

Example 3.3. (Weisfeiler [8]).

Let $R = K\{y\}$, y a differential indeterminate, R an ordinary differential ring with derivation operator ∂.

$\mathbb{G}_m$ acts on R by the regular representation $(g \cdot f)(h) = f(hg)$,

$f \in R, g, h \in \mathbb{G}_m$.

Let $N = \{1, -1\}$, the square roots of unity. N is a (normal) ∂-K-subgroup of $\mathbb{G}_m$.

The ∂-K-subalgebra R^N of differential polynomial invariants of N

is not finitely generated as a ∂-K-subalgebra. $R^N = K\{y^2, (\partial y)^2, \ldots, (\partial^n y)^2, \ldots\}$,

and cannot be written $K\{\eta_1, \ldots, \eta_n\}$, for any $\eta_1, \ldots, \eta_n \in R^N$. The proof

follows from the Ritt example cited above.

N is reductive: a difference between algebraic and differential algebraic groups.

The following is needed for the *Fundamental Theorem, part II*

of parametrized Picard-Vessiot theory.

Let G be a ∂-K-linear group, and, let R be the Hopf algebra

representing G.

G acts on R by the *regular representation*: $(g \cdot f)(h) = f(hg)$.

Let N be a normal ∂-K-subgroup of G.The set $R^N$of invariants of N

is a ∂-K-subalgebra of R.

Query: Is R^N finitely generated (as a ∂-K-algebra)?

The affirmative answer to this query is what we shall discuss today.

Remark 3.4. If $G = \mathbb{G}_m$, the Hopf algebra representing G is $R = K\left\{y, \frac{1}{y}\right\}$. $R^N = K\left\{y^2, \frac{1}{y^2}\right\}$.

Remark 3.5. The multiplication $\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ defines a comultiplication $\Delta : K\{y\} \to K\{y\} \otimes K\{y\}$:

$$\begin{aligned}\Delta(y) &= y \otimes y \\ \Delta(\partial y) &= \partial(\Delta y) = \partial y \otimes y + y \otimes \partial y \\ \Delta \circ \partial &= \partial \circ \Delta\end{aligned}$$

The inclusion $K \hookrightarrow K\{y\}$ defines a counit $\varepsilon : K\{y\} \to K :\ \varepsilon(f) = f(1)$.
Δ and ε are homomorphisms of ∂-K-algebras.

Associativity of multiplication translates into coassociativity.

of the comultiplication.

The counit satisfies a counitary property.

The comultiplication and counit define on the ∂-K-vector space $K\{y\}$ the structure of ∂-K-bialgebra, but, not of ∂-K-Hopf algebra. We now build the structure of ∂-K-Hopf algebra, beginning with ∂-K-coalgebra.

Remark 3.6. We return to the base field k.

4 ∂-k-coalgebras

Definition 4.1. A k-coalgebra C is a *∂-k-coalgebra* if

1. the underlying k-vector space is a ∂-k-vector space
2. the comultiplication $\Delta : C \to C \otimes_k C$ is a ∂-k-linear map
3. the counit $\varepsilon : C \to k$ is a ∂-k-linear map.

Recall that the following diagrams are commutative:

$$\begin{array}{cccccc} & C \otimes C \otimes C & \stackrel{\Delta\otimes 1}{\longleftarrow} & C \otimes C & \\ 1 \otimes \Delta & \uparrow & & \uparrow & \Delta \\ & C \otimes C & \stackrel{\Delta}{\longleftarrow} & C & \end{array}$$

$$\begin{array}{ccccc} k \otimes C & \stackrel{\varepsilon\otimes 1}{\longleftarrow} & C \otimes C & \stackrel{1\otimes\varepsilon}{\longrightarrow} & C \otimes k \\ & \nwarrow\sim & \uparrow\Delta & \sim\nearrow & \\ & & C & & \end{array}$$

Example 4.2. Let $C = k\{y\}_1$, y a differential indeterminate, k an ordinary differential field,with derivation operator ∂. $C = \left\{\ell = \sum_{e\in\mathbb{Z}^m_{\geq 0}} \alpha_e \partial^e y : \alpha_e \in k\right\}$.Define

$$\begin{aligned}
\Delta(y) &= y \otimes y \\
\Delta(\partial y) &= \partial(\Delta y) = \partial y \otimes y + y \otimes \partial y \\
\Delta \circ \partial &= \partial \circ \Delta
\end{aligned}$$

Define $\varepsilon(\ell) = \ell(1)$.Tracing the coassociativity diagram:

$$\begin{aligned}
(\Delta \otimes 1)(\Delta y) &= (\Delta \otimes 1)(y \otimes y) \\
&= (y \otimes y) \otimes y \\
&= y \otimes (y \otimes y) \\
&= (1 \otimes \Delta)(\Delta y)
\end{aligned}$$

Tracing the counitary diagram:

$$\begin{aligned}
(\varepsilon \otimes 1)(\Delta y) &= (\varepsilon \otimes 1)(y \otimes y) \\
&= \varepsilon(y) \otimes y \\
&= 1 \otimes y \\
&= y
\end{aligned}$$

Thus, C is a ∂-k-coalgebra.

We fix a ∂-k-coalgebra C.

Definition 4.3. Let (C, Δ, ε) and $(C', \Delta', \varepsilon')$ be ∂-k-coalgebras. A ∂-k-linear map $\varphi : C \rightarrow C'$ is a ∂-k-homomorphism if $(\varphi \otimes \varphi) \circ \Delta = \Delta'$, and, $\varepsilon = \varepsilon' \circ \varphi'$.

The kernel of φ is a ∂-k-subcoalgebra of V and the image of φ is a ∂-k-subcoalgebra of V'.

Definition 4.4. Let D be a k-subspace of C. D is a k-subcoalgebra of C if $\Delta(D) \subset D \otimes D$.

D is a ∂-k-subcoalgebra of C if, in addition, D is a ∂-k-subspace of C.

Example 4.5. Return to the ∂-k-coalgebra $C = k\{y\}_1$ of Example 4.2.

Let D be the k-subspace with basis $y, \partial y$.

$$\Delta(\alpha y + \beta \partial y) = \alpha(y \otimes y) + \beta(\partial y \otimes y + y \otimes \partial y)$$

Thus, D is a k-subcoalgebra of C, but, not a ∂-k-subcoalgebra of C

Definition 4.6. Let $X \subset C$. The *∂-k-subcoalgebra D of C generated by X* is the intersection of the family of ∂-k-subcoalgebras of C containing X.

Lemma 4.7. *Let D be a k-subcoalgebra of C. Then $\langle D \rangle$ is the* ∂-k-subcoalgebra of C generated by D.

Proof. This follows from the fact that Δ is ∂-k-linear. ☐

Example 4.8. Return to the last example. $C = k\{y\}_1$. Let D be the k-subspace with basis $y, \partial y$. D is a k-subcoalgebra of C. The ∂-k-subcoalgebra of C generated by D is $\langle y \rangle = C$.

Definition 4.9.

1. Let C be a k-coalgebra. C is *finite-dimensional* if the underlying k-space is finite-dimensional.

2. Let C be a ∂-k-coalgebra. C is *finitely generated* if the underlying ∂-k-space is finitely generated.

Corollary 4.10. *Let D be a finite dimensional k-subcoalgebra of C. The ∂-k-subcoalgebra $< D >$ is finitely generated. A basis for D is a set of generators for the ∂-k-subcoalgebra generated by D.*

Theorem 4.11. *The fundamental theorem of coalgebras (Sweedler [6]) Let C be a k-coalgebra. If $\eta \in C$, the k-subcoalgebra generated by η is finite-dimensional.*

Corollary 4.12. *Let C be a ∂-k-coalgebra. If $\eta_1, \ldots, \eta_n$ are in C, the ∂-k-subcoalgebra of C generated by $\eta_1, \ldots, \eta_n$ is finitely generated.*

Remark 4.13. The finite generating set of the ∂-k-subcoalgebra generated by $\eta_1, \ldots, \eta_n$ does not usually equal $\{\eta_1, \ldots, \eta_n\}$.

5 ∂-k-Hopf algebras

Remark 5.1. Let R be a commutative k-algebra. Then, the k-space $R \otimes_k R$ has a natural structure of commutative k-algebra. $(\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) = \alpha_1\beta_1 \otimes \alpha_2 \otimes \beta_2$. $1 \otimes 1$ is the unit. $k = k(1 \otimes 1) = k \otimes 1 = 1 \otimes k$.

Definition 5.2. Let R be a commutative k-algebra. R is a commutative k-bialgebra if

1. The underlying k-space of R is a k-coalgebra,

2. Δ and ε are k-algebra homomorphisms.

Definition 5.3. Let R be a commutative ∂-k-algebra. R is a commutative *∂-k-bialgebra* if

1. the underlying ∂- k-space of R is a ∂-k-coalgebra,

2. Δ and ε are homomorphisms of ∂-k-algebras.

Example 5.4. Let $R = k\{y\}$, k an ordinary differential field with derivation operator ∂. $k\{y\}_1$ is a ∂-k-coalgebra.We extend Δ and ε to homomorphisms from the ∂-k-algebra R to the ∂-k-algebra $R \otimes_k R$. For example,

$$\begin{aligned}
\Delta\left(y\partial^2 y - (\partial y)^2\right) &= \Delta y \partial^2 \Delta y - (\partial \Delta y)^2 \\
&= (y \otimes y)\left(\partial^2 y \otimes y + 2\partial y \otimes \partial y + y \otimes \partial^2 y\right) \\
&- (\partial y \otimes y + y \otimes \partial y)^2 \\
&= y\partial^2 y \otimes y^2 + 2(y\partial y \otimes y\partial y) \\
&- (\partial y)^2 \otimes y^2 - 2(y\partial y \otimes y\partial y) - y^2 \otimes (\partial y)^2 \\
&= y^2 \otimes \left(y\partial^2 y - (\partial y)^2\right) + \left(y\partial^2 y - (\partial y)^2\right) \otimes y^2.
\end{aligned}$$

Remark 5.5. 1. A ∂-k-subbialgebra of R must be both a ∂-k-subcoalgebra and a ∂-k-subalgebra of R.

2. If R and R' are ∂-k-bialgebras, a ∂-k-homomorphism $\varphi : R \to R'$ must be both a homomorphism of ∂-k-bialgebras and ∂-k-algebras. The kernel of φ is a ∂-k-subbialgebra of R and the image of φ is a ∂-k-subbialgebra of R'.

Definition 5.6. Let R be a k-algebra, with multiplication $\mu : R \times R \to R$, and, homomorphism $\iota : k \to R$. Suppose R is a k-bialgebra. R is a *k-Hopf algebra* if there is a k-algebra endomorphism S of R, called the *antipode*, such that $\mu \circ (S \otimes 1) \circ \Delta = \mu \circ (1 \otimes S) \circ \Delta = \iota \circ \varepsilon$, and, $\varepsilon \circ S = \varepsilon$.

If, in addition R is a ∂-k-bialgebra, and, S is a ∂-k-endomorphism of R, we call R a ∂-k-Hopf algebra.

Example 5.7. The ∂-k-bialgebra $B = k\{y\}$ of Example 5.4, and, the Weisfeiler example, is not a ∂-k-Hopf algebra. There is no antipode.

We extend the ∂-k-bialgebra B to $R = k\left\{y, \frac{1}{y}\right\}$. Define $\Delta\left(\frac{1}{y}\right) = \frac{1}{y} \otimes \frac{1}{y}$, and $\varepsilon\left(\frac{1}{y}\right) = 1$. Define $S(y) = \frac{1}{y}$,and, $S\left(\frac{1}{y}\right) = y$, and, extend S to an endomorphism of ∂-k-algebras. For example,

$$\begin{aligned} S(\partial y) &= \partial\left(S(y)\right) \\ &= -\frac{\partial y}{y^2} \end{aligned}$$

Also,

$$S(y\partial^2 y - (\partial y)^2) = -\left(\frac{y\partial^2 y - (\partial y)^2}{y^4}\right)$$

Observe: $(\mu \circ (S \otimes 1) \circ \Delta)(y) = \mu\left(\frac{1}{y} \otimes y\right) = 1 = \iota\left(\varepsilon\left(y\right)\right)$.

Example 5.8. Let $y = (y_{ij})$ be an $n \times n$ matrix of ∂- indeterminates.

Set $R = k\{y, y^{-1}\}$. We will define on R the structure of ∂-k-Hopf algebra.

Define $\Delta : R \otimes_k R$ by defining

$$\Delta(y_{ij}) = \sum_{k=1}^{n} y_{ik} \otimes y_{kj}.$$

and extending Δ to a homomorphism of ∂-k-algebras. Thus,

$$\Delta(\partial_l y_{ij}) = \partial_l(\Delta y_{ij}) = \sum_{k=1}^{n} \partial_l y_{ik} \otimes y_{kj} + \sum_{k=1}^{n} y_{ik} \otimes \partial_l y_{kj}.$$

Define $\varepsilon(y_{ij}) = 1$ if $i = j$, and $\varepsilon(y_{ij}) = 0$ if $i \neq j$, and, extend to a homomorphism of ∂-k-algebras.In particular, $\varepsilon(\partial y_{ij}) = 0$ for all i, j.

Also, $((\varepsilon \otimes 1) \circ \Delta)(y_{ij}) = \sum_{k=1}^{n} \varepsilon(y_{ik}) \otimes y_{kj} = \sum_{k=1}^{n} \varepsilon(y_{ik}) y_{kj} = y_{ij}$

Define $S(y_{ij}) = (y^{-1})_{ij}$, and extend to a homomorphism of ∂-k-algebras. In particular,

$$\begin{aligned} S(\partial_l y_{ij}) &= \partial_l(S y_{ij}) \\ &= \sum_{k=1}^{n} \partial_l((y)^{-1})_{ik} \otimes y_{kj} \\ &+ \sum_{k=1}^{n} (y)_{ik}^{-1} \otimes \partial_l y_{kj}. \end{aligned}$$

We have: $(\mu \circ (S \otimes 1) \circ \Delta)(y_{ij}) = (\mu \circ (S \otimes 1)) \left(\sum_{k=1}^{n} y_{ik} \otimes y_{kj}\right)$

$= \sum_{k=1}^{n} ((y)^{-1})_{ik} y_{kj} = \iota(\varepsilon(y_{ij}))$.

It follows that $R = k\{y, y^{-1}\}$ is a ∂-k-Hopf algebra.

Definition 5.9. Let R and R' be ∂-k-Hopf algebras. A homomorphism $\varphi : R \to R'$ of ∂-k-bialgebras is homomorphism of ∂-k-Hopf algebras. ($\Longrightarrow$ $\varphi \circ S = S \circ \varphi$.)

Remark 5.10.
1. The antipode S is an involution since R is commutative. $S \circ S = 1$. In particular, S is a ∂-k-automorphism of R.
2. $\tau \circ (S \otimes S) \circ \Delta = \Delta \circ S$.

We fix a ∂-k-Hopf algebra R.

Definition 5.11. A ∂-k-subbialgebra H of R such that $S(H) \subset H$ is called a ∂-k-Hopf subalgebra of R.

H is a ∂-k-subcoalgebra , and a ∂-k-subalgebra of R, and, is stable under the antipode.

Let $X \subset R$. The ∂-k-Hopf subalgebra generated by X is the intersection of the family of ∂-k-Hopf subalgebras containing X.

Definition 5.12. A ∂-k-Hopf algebra R is *finitely generated* if its underlying ∂-k-algebra is finitely generated. Thus, there exist $\eta_1, \ldots, \eta_n \in R$ such that $R = k\{\eta_1, \ldots, \eta_n\}$.

Lemma 5.13. *Let $\eta_1, \ldots, \eta_n$ be elements of R. The ∂-k-Hopf subalgebra they generate is finitely generated (as a ∂-k-algebra).*

Proof. Let C be the ∂-k-subcoalgebra generated by $\eta_1, \ldots, \eta_n$. Then, the ∂-k-subspace C is finitelygenerated (Corollary 4.12). Set $D = S(C)$. Since S is ∂-k-linear, D is a finitely generated ∂-k-subspace of R. Now, $\Delta(D) = \Delta(S(C)) = \tau((S \otimes S)(\Delta(C))$ (Remark 5.10, 2). So, $\Delta(D) \subset \tau(S \otimes S)(C \otimes_k C) = \tau(S(C) \otimes_k S(C)) = \tau(D \otimes D) = D \otimes D$.

Thus, $\Delta(C + D) \subset \Delta(C) + \Delta(D) \subset (C + D) \otimes_k (C + D)$. So, $C + D$ is a finitely generated ∂-k-subcoalgebra of R. $S(C) \subset D$, and, $S(D) \subset S(S(C)) = C$. So, $C + D$ is stable under the antipode.

Let $\zeta_1, \ldots, \zeta_p$ be ∂-k- space generators of the ∂-k-subcoalgebra $W = C + D$. We shall show that the ∂-k-subalgebra $H = k\{\zeta_1, \ldots, \zeta_p\}$.

We first show that H is a ∂-k-Hopf subalgebra of R. Clearly, $W \subset H$. For $i = 1, \ldots, p$, $\Delta(\zeta_i)$ is in $\Delta(W) \subset W \otimes_k W \subset H \otimes_k H$. Since R is a ∂-k-bialgebra, Δ is a homomorphism of ∂-k-algebras. $H \otimes_k H$ is a ∂-k-subalgebra of $R \otimes_k R$. Therefore, $\Delta(H) = k\{\Delta(\zeta_1), \ldots, \Delta(\zeta_p)\}$ is contained in $H \otimes_k H$. Therefore, H is a ∂-k-subbialgebra of R. Similarly, since S is a ∂-k-algebra automorphism of H, and, $S(W) \subset W$, it follows that $S(H) \subset H$. So, H is a ∂-k-Hopf subalgebra of R, and, of course, is finitely generated.

We must now show that H is the smallest ∂-k-Hopf subalgebra of R containing $\eta_1, \ldots, \eta_n$. So, let L be a ∂-k-Hopf subalgebra of R containing $\eta_1, \ldots, \eta_n$. C is the smallest ∂-k-subcoalgebra of R containing $\eta_1, \ldots, \eta_n$. Since L is a ∂-k-subcoalgebra of H containing $\eta_1, \ldots, \eta_n$, $C \subset L$. Since L is stable under the antipode, $S(C) = D \subset L$. Therefore, $\zeta_1, \ldots, \zeta_p$ are contained in L. Since H is the ∂-k-subalgebra of H generated by $\zeta_1, \ldots, \zeta_p$, $H \subset L$. $\square$

6 ∂-k-Hopf ideals

We fix a ∂-k-Hopf algebra R.

Definition 6.1. Let I be a ∂-ideal of R. Then, I is a *∂-Hopf ideal* if:

1. $\Delta(I) \subset R \otimes I + I \otimes R$.
2. $\varepsilon(I) = 0$.
3. $S(I) \subset I$.

Example 6.2. Let $R = k\left\{y, \frac{1}{y}\right\}$ be the ∂-k-Hopf algebra of Example 5.7. Recall that $\Delta(y) = y \otimes y$. $\Delta\left(\frac{1}{y}\right) = \frac{1}{y} \otimes \frac{1}{y}$, $\varepsilon(y) = 1$. $\varepsilon\left(\frac{1}{y}\right) = 1$. $S(y) = \frac{1}{y}$.

Set $I = \sqrt{\left[y\partial^2 y - (\partial y)^2\right]}$.

Claim 6.3. *I is a ∂-Hopf ideal of R.*

$$\begin{aligned}\Delta\left(y\partial^2 y - (\partial y)^2\right) &= y^2 \otimes \left(y\partial^2 y - (\partial y)^2\right) \\ &\quad + \left(y\partial^2 y - (\partial y)^2\right) \otimes y^2 \\ &\in R \otimes I + I \otimes R.\end{aligned}$$

Therefore, $\Delta(I) \subset R \otimes I + I \otimes R$.

$$\varepsilon\left(y\partial^2 y - (\partial y)^2\right) = 0.$$

$$\begin{aligned}S\left(y\partial^2 y - (\partial y)^2\right) &= -\left(\frac{y\partial^2 y - (\partial y)^2}{y^4}\right) \\ &= -\left(\frac{1}{y}\right)^4 (y\partial^2 y - (\partial y)^2).\end{aligned}$$

Therefore, $\varepsilon(I) \subset I$, *and,* $S(I) \subset I$. *So,* I *is a* ∂*-Hopf ideal of* R.

Lemma 6.4. *1. Let R and R' be ∂-k-Hopf algebras, and, let $\varphi : R \to R'$ be a ∂-k-homomorphism. Then, $\ker\varphi$ is a ∂-Hopf ideal of R, and, $im\varphi$ is a ∂-k-Hopf subalgebra of R'.*

2. Let I be a ∂-Hopf ideal of R, and, let $\pi : R \to R/I$ be the quotient homomorphism of ∂-k-Hopf algebras. Then, R/I has a unique structure of ∂-k-Hopf algebra making π a homomorphism of ∂-k-Hopf algebras. If I is a ∂-Hopf ideal of R contained in $\ker\varphi$, there is a unique homomorphism $\psi : R/I \to R'$ such that $\psi \circ \pi = \varphi$.

Definition 6.5. A ∂-k-Hopf algebra R is *∂-Noetherian* if R is a ∂-Noetherian ∂-k-algebra.

Theorem 6.6. *(The main theorem) Every ∂-k-Hopf subalgebra H of a ∂-Noetherian ∂-k-Hopf algebra R is finitely generated as a ∂-k-algebra.*

Corollary 6.7. *Every ∂-Noetherian ∂-k-Hopf algebra is ∂-finitely generated.*

Corollary 6.8. *A ∂-k-Hopf algebra is ∂-Noetherian if and only if it is ∂-finitely generated.*

Proof. Theorem 1.3, and, Corollary 6.7. $\square$

We now prove the theorem.

Set $H^+ = \ker(\varepsilon\mid_H)$, the augmentation ideal of H. Then, since ε is a surjective homomorphism of the ∂-k-algebra H onto the ∂-field k, the augmentation ideal H^+ is a prime ∂-ideal (in fact it is a maximal ideal of H). Clearly, H^+ is a ∂-Hopf ideal of H. It is easy to see that the R-module

$RH^+$is a ∂-ideal of R.

Lemma 6.9. *RH^+ is a ∂-Hopf ideal of R.*

Proof. $\Delta(RH^+) = \Delta(R)\,\Delta(H^+) \subset (R \otimes R)(H \otimes H^+ + H^+ \otimes H)$

$\subset R \otimes RH^+ + RH^+ \otimes R$.

$S(RH^+) \subset S(R)S(H^+) \subset RH^+$.

$\varepsilon(RH^+) = \varepsilon(R)\,\varepsilon(H^+) = 0$. □

Lemma 6.10. *R/RH^+ is a ∂-k-Hopf algebra.*

Proof. Lemma 6.4, 2. □

Theorem 6.11. *Cartier [3] Every commutative Hopf algebra over a field of characteristic zero is reduced.*

Corollary 6.12. *RH^+ is a radical ∂-ideal of R.*

We turn now to the proof of the main theorem.

Let $\eta_1, \ldots, \eta_n$ generate the H^+ as a radical ∂-ideal (Theorem 1.3). Let I be the radical ∂-ideal of R generated by $\eta_1, \ldots, \eta_n$. Clearly, $H^+ \subset I$. Since RH^+ is a radical ∂-ideal of R containing H^+, $I \subset RH^+$. But, I is an R-module containing H^+. Therefore, $RH^+ \subset I$. Thus, $RH^+ = \sqrt{[\eta_1, \ldots, \eta_n]}$.

Now, set L equal to the ∂-k-Hopf subalgebra generated by $\eta_1, \ldots, \eta_n$. L is finitely generated as a ∂-k-subalgebra of R (Lemma 5.13). Since $\eta_1, \ldots, \eta_n$ are in the ∂-k-Hopf subalgebra H, $L \subset H$. Thus, $L^+ \subset H^+$, and, $RL^+ \subset RH^+$. Now, $\eta_1, \ldots, \eta_n$ are in H^+. Since $\eta_1, \ldots, \eta_n$ are in L, they are in $\ker(\varepsilon\,|_L) = L^+ \subset RL^+$. Since $RH^+ = \sqrt{[\eta_1, \ldots, \eta_n]}$, and, RL^+ is a radical ∂-ideal of R containing $\kappa_1, \ldots, \kappa_s$, $RL^+ \supset \sqrt{[\eta_1, \ldots, \eta_n]} = RH^+$. Therefore, $RL^+ = RH^+$.

Theorem 6.13. *Takeuchi [7], Corollary 3.10, p. 9. Let R be a commutative*

Hopf algebra over a field k (arbitrary characteristic). The map

$$H \longmapsto RH^+$$

from the set of all Hopf k-subalgebras H of R into the set of all Hopf ideals

of R is injective.

Remark 6.14. Takeuchi proves that the image of this map is the set of so-called normal Hopf ideals of R.

Corollary 6.15. *$H = L$.*

Corollary 6.16. *H is a finitely generated ∂-k-subalgebra of R.*

This ends the proof of the main theorem. It also answers in the affirmative the question posed at the beginning of the talk. For, the differential algebra of invariants of N under the regular representation is a differential Hopf subalgebra of the differentially finitely generated Hopf algebra R representing the group G, and, therefore must be differentially finitely generated over the base differential field.

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