Differential resolvents are complete and useful.

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Abstract. Is an α -th power differential resolvent of a polynomial a differential resolvent of the minimal polynomial of every solution of the resolvent? The answer is yes. In this sense differential resolvents are complete. I proved this result in February 2004. We will choose various linear combinations over constants of the real-valued solutions of the resolvents of bi-quadratic equations to create funny-shaped ellipses I call "α -ellipses" and use the computer algebra system Mathematica to plot them. For fun, I will work backwards from the derived α -th power resolvent using Kovacic's algorithm to determine the values of α for which algebraic solutions exist. If there is time, I will use the resolvent to compute upper bounds on the number of extrema on these α -ellipses.

Section 1. Differential resolvents are complete

Definition 1.1. For any positive integer m define $[m] \equiv \{i \in \mathbb{Z} \ni 1 \le i \le m\}$.

Definition 1.2. Let $\{e_i\}_{i=1}^N$ be elements of an ordinary differential field $\mathbb{F}$ with a derivation $\frac{d}{dz}$ and subfield of constants $\Bbbk$. Consider $\{e_i\}_{i=1}^N$ to be functions of the complex independent variable $z = x + \sqrt{-1} \cdot y$, with $x = \text{Re}(z)$ and $y = \text{Re}(z)$, which is defined to satisfy $\frac{d}{dz}(z) = 1$.

Definition 1.3. Define $\mathbb{Q} < e_1, ..., e_N >$ to be the smallest differential field of characteristic zero containing $\{e_i\}_{i=1}^N$. From now on, assume $\mathbb{F} = \mathbb{Q} < e_1, ..., e_N > (\Bbbk)$.

Definition 1.4. Define the monic polynomial $P(t) \equiv \sum_{i=0}^{N} (-1)^{N-i} e_{N-i} \cdot t^i$.

Definition 1.5. Define $\{r_i\}_{i=1}^n$ to be the n distinct roots of $P(t)$, which will be our complex dependent variables. Let $r = s + \sqrt{-1} \cdot \sigma$ with $s = \text{Re}(r)$ and $\sigma = \text{Im}(r)$ represent any root of $P(r) = 0$.

So e_i is the i -th elementary symmetric function of the roots of P counted with multiplicities.

Definition 1.6. Let α be an *indeterminate* or *real constant* over $\mathbb{F}$. Define an *α -power of the root* r_k to be any nonzero solution of $\frac{1}{r_k^\alpha}\frac{dr_k^\alpha}{dz} = \alpha \frac{1}{r_k}\frac{dr_k}{dz}$.

Definition 1.7. Let Λ be an element algebraic over $\mathbb{F} < r_1,...,r_n >$. Define $P_\Lambda(t) \in \mathbb{F}[t]$ to be the minimal polynomial over $\mathbb{F}$ for Λ. Define $L \equiv \deg_t P_\Lambda(t)$. Define $\{\Lambda_k\}_{k=1}^L$ to be the roots of $P_\Lambda(t)$ (not necessarily distinct) with $\Lambda_L = \Lambda$.

We assume in computations that Λ is specified by its minimal polynomial, $\psi(t)$, over $\mathbb{F} < r_1,...,r_n >$ and *not* by $P_\Lambda(t)$. There exists a linear differential equation $\Re$, which satisfies the following definition.

Definition 1.8. We call $\Re$ a Λ*-resolvent* of P if

1) $\Re\Lambda_k = 0$ for all roots $\{\Lambda_k\}_{k=1}^L$ of P_Λ

2) all terms of $\Re$ lie in $\mathbb{F}$

3) not all terms of $\Re$ are zero.

Special case of Definition 1.8. Let $q = \dfrac{l}{m}$ with $l \in \mathbb{Z}$, $m \in \mathbb{Z}^\#$, and $\gcd(l,m) = 1$. Pick a root r_k. Let $\Lambda = r_k^q$. Then $P_\Lambda(t)$ either equals or divides $\rho(t) \equiv \prod_{i=1}^n (t^m - r_i^l)$. The roots of $\rho(t)$ are $\{\Lambda_k\}_{k=1}^{m\cdot n} = \{\omega^i \cdot r_j^q\}_{\substack{i\in[m]\\ j\in[n]}}$ where $\omega \equiv \exp\left(\dfrac{2\pi\sqrt{-1}}{m}\right) \notin \mathbb{R}$ for $m > 2$. Therefore all the roots of $P_\Lambda(t)$ are constants times multiples of q - powers of the roots of P. Therefore, we may identify Λ with the power q.

Definition 1.9. When $\Lambda = r_k^q$ in Definition 1.8 with rational power q, we call a Λ-resolvent $\Re$ of P a q*-resolvent* of P.

We separate out the special case of rational q because in computer computations numbers can be only rational. When $\Lambda = r_k^\alpha$ for *indeterminate or non-rational* α, *there exists no minimal polynomial for* Λ *over* $\mathbb{F}$. Nevertheless, a differential resolvent for Λ exists if we make the following modification to Definition 1.8.

Definition 1.10. If we specialize $\Lambda_k = r_k^{\alpha}$, $L = n$ and define $P_\Lambda \equiv \prod_{i=1}^{n}(t - r_i^{\alpha})$ in Definition 1.8 and require the terms of $\Re$ lie in $\mathbb{F}[\alpha]$, we call a Λ-resolvent $\Re$ of P an α*-resolvent* of P.

Definition 1.11. Let $\vec{c} = (c_1,...,c_L) \in \Bbbk^L$. Define $w_{\vec{c}} \equiv \sum_{k=1}^{L} c_k \cdot \Lambda_k$. Define $P_{\vec{c}}(t) \in \mathbb{F}[t]$ to be the minimal polynomial of $w_{\vec{c}}$ over $\mathbb{F}$.

Then $P_{\vec{c}}$ exists, and $w_{\vec{c}}$ is a solution of the Λ-resolvent given by Definition 1.8.

Definition 1.12. Define $w \equiv \sum_{i=1}^{n} c_i \cdot r_i^{\alpha}$ or $w \equiv \sum_{i=1}^{n} c_i \cdot r_i^{q}$ to be *any* linear combination of α-powers or q-powers, respectively, of the roots over constants $(c_1,...,c_L) \in \Bbbk^L$.

Then w exists and is a solution of the α-resolvent of Definition 1.10 or the q-resolvent of Definition 1.9, respectively. Therefore w can be real-valued when $r_i \geq 0$, $c_i \in \mathbb{R}, \forall i \in [n]$, $\alpha \in \mathbb{R}$, or complex but single-valued when $c_i \in \mathbb{C}, \forall i \in [n]$, $\alpha \in \mathbb{Z}$, or multi-valued and transcendental over $\mathbb{C}$ when c_i and α are transcendental constants over $\mathbb{C}$.

Question. Is $\Re$ a resolvent of $P_{\vec{c}}(t)$? In other words, is *every* root of $P_{\vec{c}}(t)$ given by Definition 1.11 a linear combination of the roots of $P_\Lambda(t)$ over $\Bbbk$? The answer is yes. In fact, we may prove the following stronger result.

Theorem 1.13. Every root of $P_{\vec{c}}(t)$ (Definition 1.11) is a linear combination of the roots of $P_\Lambda(t)$ (Definition 1.7) over the (smaller) constant subring $\mathbb{Z}[c_1,...,c_L]$ and is therefore a solution of $\Re$.

Proof. Let $\{\xi_i\}_{i=1}^{L}$ be indeterminates. Define $\phi_1(t;\xi_1,...,\xi_L) \equiv t - \sum_{i=1}^{L} c_i \cdot \xi_i$, a polynomial in ξ_1. Define $\phi_2(t;\xi_2,...,\xi_L)$ to be the resultant of $\phi_1(t;\xi_1,...,\xi_L)$ and $P_\Lambda(\xi_1)$ eliminating ξ_1. By definition of resultant $\phi_2(t;\xi_2,...,\xi_L) = \prod_{k=1}^{L} \phi_1(t;\Lambda_k,\xi_2,...,\xi_L)$ with $\deg_t \phi_2(t;\xi_2,...,\xi_L) = L$. Thus, the t-roots of

$\phi_2(t;\xi_2,...,\xi_L)$ are those t which make $t-\sum_{i=2}^{L} c_i \cdot \xi_i - c_1 \cdot \Lambda_k = 0$ for any $k \in [L]$. In other words, all roots of $\phi_2(t;\xi_2,...,\xi_L)$ are linear combinations over constants of $\{\Lambda_k\}_{k=1}^{L}$ and $\{\xi_i\}_{i=2}^{L}$.

Now we eliminate the ξ_i's one by one, replacing them with the roots of P_Λ. Recursively define $\phi_{m+1}(t;\xi_{m+1},...,\xi_L)$ for $m \in [L]$ to be the resultant of $\phi_m(t;\xi_m,...,\xi_L)$ and $P_\Lambda(\xi_m)$ eliminating ξ_m where $\phi_{L+1} = \phi_{L+1}(t)$ does not involve $\{\xi_i\}_{i=1}^{L}$. By definition of resultant

$\phi_{m+1}(t;\xi_{m+1},...,\xi_L) = \prod_{k=1}^{L} \phi_m(t;\Lambda_k,\xi_{m+1},...,\xi_L)$ with $\deg_t \phi_{m+1}(t;\xi_{m+1},...,\xi_L) = L^m$ and *most importantly* $\phi_{m+1}(t;\xi_{m+1},...,\xi_L) \in \mathbb{F}[t;\xi_{m+1},...,\xi_L]$. Thus, the t-roots of $\phi_{m+1}(t;\xi_{m+1},...,\xi_L)$ are the t-roots of $\phi_m(t;\Lambda_k,\xi_{m+1},...,\xi_L)$ for any $k \in [L]$. But all the t-roots of $\phi_m(t;\xi_m,\xi_{m+1},...,\xi_L)$ equal $\sum_{k=1}^{m-1} c_k \cdot \Lambda_{J_k^{(m-1)}} + \sum_{k=m}^{L} c_k \cdot \xi_k$ for each $(m-1)$-tuple $J^{(m-1)} = (j_1,...,j_{m-1}) \in [L]^{m-1}$ where $J_k^{(m-1)} = j_k$ is the k-th component of $J^{(m-1)}$. Thus, the t-roots of $\phi_{m+1}(t;\xi_{m+1},...,\xi_L)$ equal $\sum_{k=1}^{m} c_k \cdot \Lambda_{J_k^{(m)}} + \sum_{k=m+1}^{L} c_k \cdot \xi_k$. Finally we arrive at $\phi_{L+1}(t)$, all of whose t-roots equal $\sum_{k=1}^{L} c_k \cdot \Lambda_{J_k}$ where $J = (j_1,...,j_L) \in [L]^L$, with $\deg_t \phi_{L+1}(t) = L^L$ and $\phi_{L+1}(t) \in \mathbb{F}[t]$.

Since $\mathbb{F}$ is a perfect field, and $P_{\vec{c}}(t)$ and $\phi_{L+1}(t)$ share the root $w_{\vec{c}}$ and lie in $\mathbb{F}[t]$, by properties of a minimal polynomial $P_{\vec{c}}(t)$ divides $\phi_{L+1}(t)$ in $\mathbb{F}[t]$. Hence all the roots of $P_{\vec{c}}(t)$ are contained in the set of roots of $\phi_{L+1}(t)$. Therefore, all roots of $P_{\vec{c}}(t)$ are linear combinations of $\{\Lambda_k\}_{k=1}^{L}$ over the constant ring $\mathbb{Z}[c_1,...,c_L]$. □

Section 2. The α-resolvent for a general quadratic.

Definition 2.1. Let ' denote $\frac{d}{dz}$. Define $r_1(z)$ and $r_2(z)$ to be the two distinct roots of the *general quadratic* $P(r) \equiv r^2 - e_1 \cdot r + e_2 = 0$ for which there may or may not exist relations over $\Bbbk$ (Definition 1.2) of $e_1, e_2, e_1', e_2', \ldots$ except $e_1^2 - 4e_2 \neq 0$.

Definition 2.2. Define $\Delta \equiv (r_1 - r_2)^2$ and call it the *discriminant* of P.

Definition 2.3. Define W to be $\frac{\alpha \cdot e_2^{\alpha-1}}{\sqrt{\Delta}}$ times the *Wronskian* $\det\begin{bmatrix} r_1^\alpha & r_2^\alpha \\ (r_1^\alpha)' & (r_2^\alpha)' \end{bmatrix}$.

Then the *monic* second-order homogeneous α-resolvent over the differential field $\mathbb{Q} < e_1, e_2 >$ of the general quadratic is

$$\frac{d^2w}{dz^2} + (f_{0,1} + f_{1,1} \cdot \alpha) \cdot \frac{dw}{dz} + (f_{1,0} \cdot \alpha + f_{2,0} \cdot \alpha^2) \cdot w = 0 \tag{2.1}$$

with

$$f_{0,1} = -\frac{W'}{W} + \frac{1}{2}\frac{\Delta'}{\Delta} + \frac{e_2'}{e_2}\ ,\ f_{1,1} = -\frac{e_2'}{e_2},\ f_{1,0} = \frac{1}{2}\left(\frac{e_2'}{e_2} \cdot \left(\frac{W'}{W} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) - \frac{e_2''}{e_2}\right),$$

$$f_{2,0} = \frac{1}{2}\left(\frac{W'}{W}\frac{\Delta'}{\Delta} - \frac{\Delta''}{\Delta}\right) - f_{1,0} \tag{2.2}$$

This monic form of the α-resolvent of P is useful for computations.

The second-order homogeneous α-resolvent of P with terms in the ring $\mathbb{Z}[z]$ and no common factor in $\mathbb{Z}[z]$ except ± 1 (called the *Cohnian*) has the form

$$F_{0,2} \cdot \frac{d^2w}{dz^2} + (F_{0,1} + F_{1,1} \cdot \alpha) \cdot \frac{dw}{dz} + (F_{1,0} \cdot \alpha + F_{2,0} \cdot \alpha^2) \cdot w = 0 \tag{2.3}$$

where we compute the *coefficient functions* $F_{i,m} \in \mathbb{Z}[z]$ using the author's *powersum formula* [7] and then factor. In fact $F_{0,2} = 4W \cdot \Delta \cdot e_2$, $F_{0,1} = -4W' \cdot \Delta \cdot e_2 + 2W \cdot \Delta' \cdot e_2 + 4W \cdot \Delta \cdot e_2'$, $F_{1,1} = -4W \cdot \Delta \cdot e_2'$, $F_{1,0} = 2W' \cdot \Delta \cdot e_2' - W \cdot \Delta' \cdot e_2' - 2W \cdot \Delta \cdot e_2''$, $F_{2,0} = 2\left(W' \cdot \Delta' \cdot e_2 - W \cdot \Delta'' \cdot e_2\right) - F_{1,0}$ up to a multiple in $\mathbb{Z}$.

Definition 2.4. $R_2(z) \equiv 4W \cdot \Delta \cdot e_2$, $R_1(z,\alpha) \equiv F_{0,1} + F_{1,1} \cdot \alpha$, and $R_0(z,\alpha) \equiv F_{1,0} \cdot \alpha + F_{2,0} \cdot \alpha^2$.

Later we will suppress the letter α in $R_1(z,\alpha)$ and $R_2(z,\alpha)$ and write the α-resolvent as just

$$R_2(z)\cdot\frac{d^2w}{dz^2}+R_1(z)\cdot\frac{dw}{dz}+R_0(z)\cdot w=0\,. \tag{2.4}$$

Example 2.5. Computing Taylor series. An easy computation with a 3×3 Wronskian type determinant shows there exist m-th order resolvents for quadratic equations $r^2-e_1(z)\cdot r+e_2(z)=0$ with roots $\{r_k\}_{k=1}^2$ of the form

$$F_{0,m}\cdot\frac{d^m w}{dz^m}+\left(\sum_{i=0}^{m-1}F_{i,1}\cdot\alpha^i\right)\cdot\frac{dw}{dz}+\alpha\cdot\left(\sum_{i=0}^{m-1}F_{i+1,0}\cdot\alpha^i\right)\cdot w=0\,. \tag{2.5}$$

We may (at least theoretically) use the *powersum formula* $F_{i,k}=\det\left[q^{i'}\frac{d^{k'}p_q}{dz^{k'}}\right]_{\substack{q\times(i',k')\\(i',k')\neq(i,k),1\le q\le 2m}}$ [7] to compute the $2m+1$ unknown coefficient functions $F_{i,k}\in\mathbb{Z}\{e_1,e_2\}$ where $p_q\equiv r_1^q+r_2^q\in\mathbb{Z}[e_1,e_2]$ is the q-th *powersum*. We can then (at least theoretically) express w as a Taylor series depending linearly upon the two arbitrary constants $\left.\frac{dw}{dz}\right|_{z=z_0}$ and $\left.w\right|_{z=z_0}$. Is this a computationally feasible problem?

It is worth mentioning that *only for* $\alpha\in\mathbb{Z}$ does there exist an *inhomogeneous* first-order linear ordinary differential resolvent $\frac{dr^\alpha}{dz}=\Sigma_1\cdot r+\Sigma_0$, where a similar powersum formula gives

$$\Sigma_1=\det\begin{bmatrix}p_\alpha{}' & 2\\ \frac{\alpha}{\alpha+1}p_{\alpha+1}{}' & p_1\end{bmatrix}\div\det\begin{bmatrix}p_\alpha & 2\\ p_{\alpha+1} & p_1\end{bmatrix}\text{ and}$$

$$\Sigma_0=\det\begin{bmatrix}p_\alpha & p_\alpha{}'\\ p_{\alpha+1} & \frac{\alpha}{\alpha+1}p_{\alpha+1}{}'\end{bmatrix}\div\det\begin{bmatrix}p_\alpha & 2\\ p_{\alpha+1} & p_1\end{bmatrix} \tag{2.6}$$

for $\alpha\neq -1$. In Section 10 we will use the inhomogeneous resolvent

$$\frac{dr}{dz}=\frac{1}{2}\frac{\Delta'}{\Delta}\cdot r+\frac{W}{\Delta} \tag{2.7}$$

with $\Delta,W\in\mathbb{Q}[z]$ given by Definitions 2.2 & 2.3 to determine the regions of convexity and concavity of the α-powers of the roots.

Section 3. The α -resolvent for a particular bi-quadratic.

We will now focus on the real-valued loci of quadratic polynomials in 2 variables over integer constants. Consider the 3-dimensional conic section

$$A\cdot z^2 + B\cdot z\cdot r + C\cdot r^2 + D\cdot z + E\cdot r + F = 0 \tag{3.1}$$

in the notation of page 130 of [1] with $A,B,C,D,E,F \in \mathbb{Z}$ and $z,r \in \mathbb{C}$.

Definition 3.1. Define $\pi_2 \equiv 4AC - B^2 > 0$, $\pi_1 \equiv BE - 2CD$, $\pi_0 \equiv E^2 - 4CF$, $\rho_1 \equiv BD - 2AE$, and $\rho_0 \equiv 2BF - DE$.

One can determine from Definitions 2.2, 2.3 and 3.1 that $C\cdot e_1 = -(Bz+E)$, $C\cdot e_2 = Az^2 + Dz + F$, $C^2\Delta = -\pi_2\cdot z^2 + 2\pi_1\cdot z + \pi_0$, and $C^2 W = C^2 \det\begin{bmatrix} 1\cdot e_1 & 2\cdot e_2 \\ e_1{}' & e_2{}' \end{bmatrix} = \rho_1\cdot z + \rho_0$. The real locus of equation (3.1) is always an ellipse ($\pi_2 > 0$), hyperbola ($\pi_2 < 0$), parabola ($\pi_2 = 0$), a single point, two straight lines, or the empty set.

Definition 3.2. Define γ_1 and γ_2 to be the two distinct roots of $\Delta(z)$.

Then $\gamma_1 \equiv \dfrac{\pi_1 - \sqrt{\pi_1^2 + \pi_0\pi_2}}{\pi_2}$ and $\gamma_5 \equiv \dfrac{\pi_1 + \sqrt{\pi_1^2 + \pi_0\pi_2}}{\pi_2}$ when $\pi_2 \neq 0$. Choose r_1 and r_2 in Definition 2.1 such that $r_1 \le r_2$ for all real values of z which make r real, namely the real domain $\gamma_1 \le z \le \gamma_5$.

Our running example from this point on will be the bi-quadratic

$$52369\cdot z^2 + 73920\cdot z\cdot r + 436041\cdot r^2 - 1940670\cdot z - 5469210\cdot r + 25834841 = 0 \tag{3.2}$$

so $A = 52369$, $B = 73920$, $C = 436041$, $D = -1940670$, $E = -5469210$, and $F = 25834841$. One can verify that $\pi_2 > 0$, so the real-valued locus of this equation is an ellipse.

Observe that since $F_{0,2}$ is the leading term of the resolvent (2.3), the linear factor in z is a rational number times the Wronskian, and the third factor is a rational number times the discriminant, it follows that the middle quadratic factor provides two *apparent singularities* of resolvent (2.3).

$$F_{0,2} = (2318581 - 3\cdot 13^2\cdot 17^2\cdot z)\cdot(71\cdot 363871 - 2\cdot 3^2\cdot 5\cdot 21563\cdot z + 52369\cdot z^2)\cdot$$
$$(2^3\cdot 1076903 - 2\cdot 3\cdot 5\cdot 13^2\cdot 17^2\cdot z + 13^2\cdot 17^2\cdot z^2)$$

$$F_{0,1} = -3\cdot 3011\cdot 5539330887142411 + 2\cdot 47\cdot 111617017\cdot 1156492681\cdot z$$
$$-3\cdot 13^2\cdot 17^2\cdot 1202347\cdot 6577313\cdot z^2 + 2^3\cdot 3\cdot 13^2\cdot 17^2\cdot 41413122083\cdot z^3$$
$$-2\cdot 3\cdot 13^4\cdot 17^4\cdot 52369\cdot z^4$$

$$F_{1,1} = 2(-3^2\cdot 5\cdot 21563 + 52369\cdot z)\cdot(-2318581 + 3\cdot 13^2\cdot 17^2\cdot z)\cdot$$
$$(2^3\cdot 1076903 - 2\cdot 3\cdot 5\cdot 13^2\cdot 17^2\cdot z + 13^2\cdot 17^2\cdot z^2)$$

$$F_{1,0} = -19\cdot 379\cdot 204059392278541 + 3^3\cdot 5\cdot 13^2\cdot 17^2\cdot 13187\cdot 3135037\cdot z$$
$$-13^2\cdot 17^2\cdot 52369\cdot 6835007\cdot z^2 + 3\cdot 13^4\cdot 17^4\cdot 52369\cdot z^3$$

$$F_{2,0} = (5^2\cdot 257\cdot 797\cdot 110933 - 2\cdot 3\cdot 5\cdot 13^2\cdot 17^2\cdot 52369\cdot z + 13^2\cdot 17^2\cdot 52369\cdot z^2)\cdot$$
$$(2318581 - 3\cdot 13^2\cdot 17^2\cdot z) \quad . \qquad (3.3)$$

We write the integral coefficients of the α-resolvent in factored form in (3.3) to aid checking of calculations when we apply Kovacic's algorithm in Section 11 to verify that resolvent (3.3) has algebraic solutions if and only if $\alpha \in \mathbb{Q}$.

Example 3.3. When α is a rational number, we use the letter q in its place. For example, when $q = -\frac{5}{2}$ in (2.4) we have $w = c_1\cdot r_1^{-5/2} + c_2\cdot r_2^{-5/2}$ and

$$R_2 = 4\cdot(-2318581 + 146523\cdot z)\cdot(8615224 - 1465230\cdot z + 48841\cdot z^2)\cdot$$
$$(25834841 - 1940670\cdot z + 52369\cdot z^2)$$

$$R_1 = 4\cdot 7\cdot(-3\cdot 11\cdot 11027\cdot 57689741225999 + 2^4\cdot 3\cdot 31\cdot 4663\cdot 822961484621\cdot z$$
$$-2^2\cdot 3^4\cdot 13^2\cdot 17^2\cdot 211\cdot 169849051\cdot z^2 + 2^2\cdot 13^2\cdot 123674828429\cdot z^3$$
$$-3\cdot 13^4\cdot 17^4\cdot 52369\cdot z^4)$$

$$R_0 = 3\cdot 5\cdot 7\cdot(24166991\cdot 18766839137 - 3\cdot 5\cdot 13^2\cdot 17^2\cdot 103\cdot 1168019539\cdot z$$
$$+3\cdot 13^2\cdot 17^2\cdot 52369\cdot 749863\cdot z^2 - 13^4\cdot 17^4\cdot 52369\cdot z^3) \quad . \qquad (3.5)$$

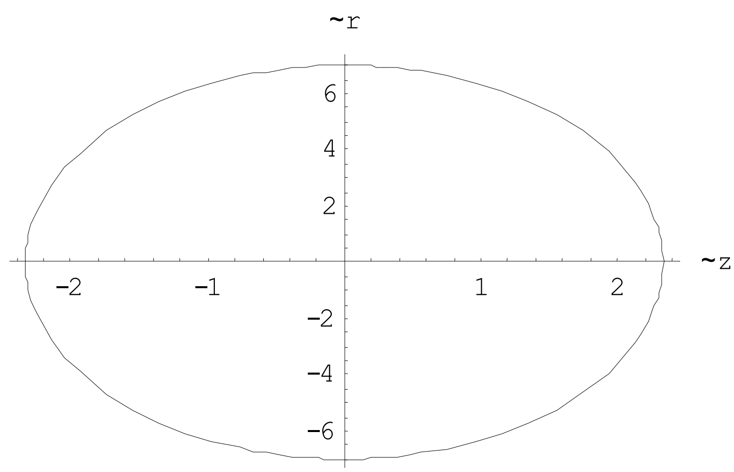
We now wish to see what the graphs of two solutions of resolvent (3.3) look like when forced to match at their critical points.

Section 4. Rotation & Translation of the Ellipse

Initial ellipse 4.1. Consider the bi-quadratic

$$9 \cdot \underset{\sim}{z}^2 + 1 \cdot \underset{\sim}{r}^2 - 49 = 0, \tag{4.1}$$

an equation of the form $\underset{\sim}{A} \cdot \underset{\sim}{z}^2 + \underset{\sim}{B} \cdot \underset{\sim}{z} \cdot \underset{\sim}{r} \cdot + \underset{\sim}{C} \cdot \underset{\sim}{r}^2 + \underset{\sim}{D} \cdot \underset{\sim}{z} + \underset{\sim}{E} \cdot \underset{\sim}{r} + \underset{\sim}{F} = 0$ with $\underset{\sim}{B} = \underset{\sim}{D} = \underset{\sim}{E} = 0$. The real-valued locus of equation (4.1) looks like



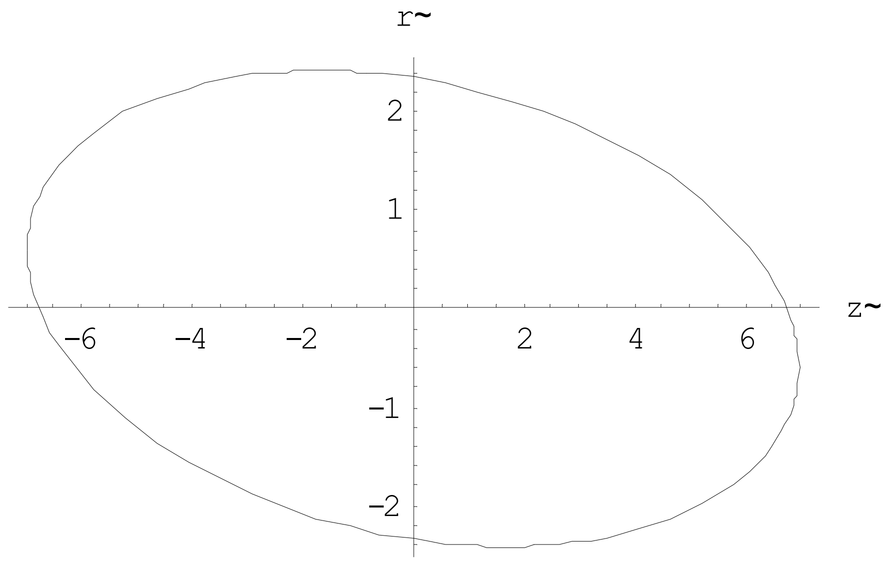
Rotated ellipse 4.2. First multiply equation (4.1) by 48841 to get $439569 \cdot \underset{\sim}{z}^2 + 48841 \cdot \underset{\sim}{r}^2 = 2393209$. Let us use the Pythagorean triplet $21^2 + 220^2 = 221^2$ to construct an angle of rotation so that the quadratic equations for both the initial ellipse and the rotated ellipse will have integral coefficients. Let $\begin{bmatrix} \underset{\sim}{z} \\ \underset{\sim}{r} \end{bmatrix} = \begin{bmatrix} \frac{21}{221} & \frac{220}{221} \\ -\frac{220}{221} & \frac{21}{221} \end{bmatrix} \cdot \begin{bmatrix} \tilde{z} \\ \tilde{r} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \tilde{z} \\ \tilde{r} \end{bmatrix} = \begin{bmatrix} \frac{21}{221} & -\frac{220}{221} \\ \frac{220}{221} & \frac{21}{221} \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{z} \\ \underset{\sim}{r} \end{bmatrix}$. Therefore (see Appendix for notation) $\tau \equiv \frac{\tilde{C} - \tilde{A}}{\tilde{B}} = \frac{436041 - 52369}{73920} = \frac{47959}{9240}$, $\tan\theta = -\frac{21}{220}$,

$\underset{\sim}{C} = (\tau - \sqrt{1+\tau^2}) \cdot \frac{\tilde{B}}{2} + \tilde{A} = 48841$, $\underset{\sim}{A} = (\tau + \sqrt{1+\tau^2}) \cdot \frac{\tilde{B}}{2} + \tilde{A} = 439569$.

Then $\tilde{A} = 52369$, $\tilde{B} = 73920$, $\tilde{C} = 436041$, $\tilde{D} = 0$, $\tilde{E} = 0$, $\tilde{F} = -2393209$ so

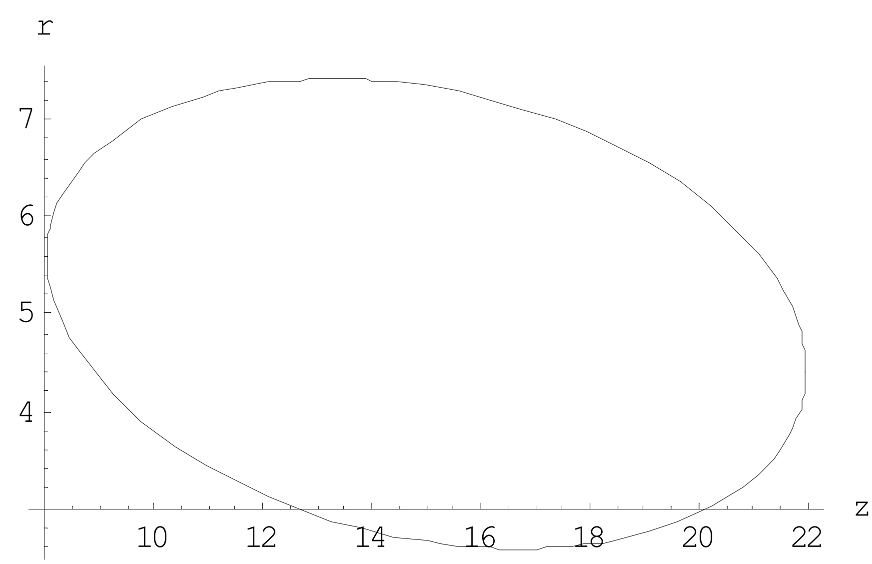
$$52369 \cdot \tilde{z}^2 + 73920 \cdot \tilde{z} \cdot \tilde{r} + 436041 \cdot \tilde{r}^2 = 2393209. \tag{4.2}$$

The real-valued locus of equation (4.2) looks like



The new axes $(\underset{\sim}{z},\underset{\sim}{r})$ are rotated clockwise by the angle $\theta = -\arctan\frac{21}{220} \approx -9.51662$ radians $\approx -5.45262°$, and thus rotated counterclockwise by 9.51662 radians.

Shifted ellipse 4.3. If we replace $\tilde{z}$ with $z-15$ and $\tilde{r}$ with $r-5$ equation (4.2), we get equation (3.2) whose real-valued locus looks like



. We made this shift in order to get all positive r-values whenever $\text{Im}(r) = 0$ since we will want $r^{\alpha} \in \mathbb{R}\ \forall \alpha \in \mathbb{R}$.

Section 5. Creating closed smooth curves by distorting the ellipse (4.3) by the power α

The two roots of equation (3.1) satisfy

$r^2 - 2(\eta_1 \cdot z + \eta_2)\cdot r + (\eta_1^2 + \eta_3^2)\cdot z^2 + 2(\eta_1 \cdot \eta_2 - \eta_3^2 \cdot \gamma_3)\cdot z + (\eta_2^2 + \eta_3^2 \cdot \gamma_1 \cdot \gamma_5) = 0$ where $\frac{A}{C} = \eta_1^2 + \eta_3^2$,

$\frac{B}{C} = -2\eta_1$, $\frac{D}{C} = 2(\eta_1 \cdot \eta_2 - \eta_3^2 \cdot \gamma_3)$, $\frac{E}{C} = -2\eta_2$, $\frac{F}{C} = \eta_2^2 + \eta_3^2 \cdot \gamma_1 \cdot \gamma_5$. The inverse relations are

$$\eta_1 = -\frac{B}{2C},\ \eta_2 = -\frac{E}{2C},\ \eta_3 = \frac{\sqrt{\pi_2}}{2C}. \tag{5.1}$$

Therefore we may write the roots of (3.1) as $r_1 = \eta_1 \cdot z + \eta_2 - \eta_3 \cdot \sqrt{(z-\gamma_1)\cdot(\gamma_5 - z)}$ and

$r_2 = \eta_1 \cdot z + \eta_2 + \eta_3 \cdot \sqrt{(z-\gamma_1)\cdot(\gamma_5 - z)}$ with $\eta_k, \gamma_k \in \overline{\mathbb{Q}}$ and $\eta_3, \gamma_1, \gamma_5 \geq 0$.

Definition 5.1. Define $\tilde{\gamma} \equiv \frac{\gamma_5 - \gamma_1}{2}$ and $\gamma_3 \equiv \frac{\gamma_1 + \gamma_5}{2}$.

In order to keep $r_1(x) \geq 0$ for all x in the interval $[\gamma_1, \gamma_5]$ so that r_1^α is real for any $\alpha \in \mathbb{R}$ it was sufficient that $\eta_1 \cdot \gamma_1 + \eta_2 = \min_{x\in[\gamma_1,\gamma_5]} (\eta_1 \cdot x + \eta_2) \geq \max_{x\in[\gamma_1,\gamma_5]} \eta_3 \cdot \sqrt{(x-\gamma_1)\cdot(\gamma_5 - x)} = \eta_3 \cdot \tilde{\gamma}$. Thus, it was sufficient to choose $\eta_2 \geq \eta_3 \cdot \tilde{\gamma} - \eta_1 \cdot \gamma_1$. One can verify that equation (3.2) satisfies this condition.

Which linear combinations $w_\beta = \beta_1 \cdot r_1^\alpha + \beta_2 \cdot r_2^\alpha$ and $w_\lambda = \lambda_1 \cdot r_1^\alpha + \lambda_2 \cdot r_2^\alpha$ satisfy $\lim_{x\to\gamma_1} \frac{dw_\beta}{dx} = -\infty$, $\lim_{x\to\gamma_5} \frac{dy_\beta}{dx} = +\infty$, $\lim_{x\to\gamma_1} \frac{dy_\lambda}{dx} = +\infty$, $\lim_{x\to\gamma_5} \frac{dy_\lambda}{dx} = -\infty$, and $w_\beta(\gamma_1) = w_\lambda(\gamma_1) \neq \pm\infty$ and $w_\beta(\gamma_5) = w_\lambda(\gamma_5) \neq \pm\infty$?

In other words, when do they form a continuous curve without a cusp?

The condition $w_\beta(\gamma) = w_\lambda(\gamma)$ is equivalent to $\beta_1 + \beta_2 = \lambda_1 + \lambda_2$. We have $r_1 \leq r_2$ whenever $\Delta \geq 0$ where we choose γ_1, γ_5 such that $\lim_{x\to\gamma_1} \frac{dr_1}{dx} = -\infty$ $\lim_{x\to\gamma_1} \frac{dr_2}{dx} = +\infty$ $\lim_{x\to\gamma_5} \frac{dr_1}{dx} = +\infty$ $\lim_{x\to\gamma_5} \frac{dr_2}{dx} = -\infty$ with $\gamma_1 < \gamma_5$ for an ellipse and $\gamma_1 > \gamma_5$ for a hyperbola. Since $\lim_{x\to\gamma} \frac{r_1}{r_2} = 1$ and $\lim_{x\to\gamma} \frac{dr_1/dx}{dr_2/dx} = -1$ it follows that

$\lim_{x\to\gamma} \frac{dw_\beta}{dx} = \alpha \cdot \lim_{x\to\gamma}(\beta_1 \cdot \left(\frac{r_1}{r_2}\right)^{\alpha-1} \cdot \frac{dr_1/dx}{dr_2/dx} + \beta_2) \cdot \lim_{x\to\gamma} r_2^{\alpha-1} \cdot \frac{dr_2}{dx} = \alpha \cdot (\beta_2 - \beta_1)\cdot (r_2(\gamma))^{\alpha-1} \cdot \lim_{x\to\gamma} \frac{dr_2}{dx}$. Similarly

$\lim_{x\to\gamma}\frac{dw_\lambda}{dx}=\alpha\cdot(\lambda_2-\lambda_1)\cdot(r_2(\gamma))^{\alpha-1}\cdot\lim_{x\to\gamma}\frac{dr_2}{dx}$. Therefore, $-\lim_{x\to\gamma}\frac{dw_\beta}{dx}=\infty=\lim_{x\to\gamma}\frac{dw_\lambda}{dx}$ if we choose $(\beta_2-\beta_1)\cdot(\lambda_2-\lambda_1)<0$. Combined with the condition $\beta_1+\beta_2=\lambda_1+\lambda_2$ we may create many closed ellipse-like curves. We will choose $\beta_1=\lambda_2$ and $\beta_2=\lambda_1$ since the weaker condition $\{(\beta_2-\beta_1)\cdot(\lambda_2-\lambda_1)<0$ and $\beta_1+\beta_2=\lambda_1+\lambda_2\}$ generates ellipses of no greater complexity.

Section 6. Approximate bounds on w for plotting

We may use the extrema of r_1^α and r_2^α separately to give bounds on w_λ. Then $\frac{dr_2}{dx}=\eta_1+\eta_3\cdot\frac{-x+2\gamma_3}{\sqrt{(x-\gamma_1)\cdot(\gamma_5-x)}}$ and $\frac{d^2r_2}{dx^2}=-\eta_3\cdot\frac{(x-2\gamma_3)^2+(x-\gamma_1)\cdot(\gamma_5-x)}{\sqrt{((x-\gamma_1)\cdot(\gamma_5-x))^3}}<0,\forall x\in[\gamma_1,\gamma_5]$. Then r_2 reaches a maximum value of $(\eta_1\cdot\gamma_3+\eta_2)+\sqrt{\eta_1^2+\eta_3^2}\cdot\tilde{\gamma}$ at $\gamma_2\equiv\gamma_3+\frac{\eta_1}{\sqrt{\eta_1^2+\eta_3^2}}\cdot\tilde{\gamma}$. The other root of the equation $\frac{dr_2}{dx}=0\Rightarrow 0=\eta_1^2\cdot(x-\gamma_1)\cdot(\gamma_5-x)-\eta_3^2\cdot(2\gamma_3-x)^2$ is $\gamma_4\equiv\gamma_3-\frac{\eta_1}{\sqrt{\eta_1^2+\eta_3^2}}\cdot\tilde{\gamma}$. For equation (3.2), $\eta_1<0$ so $\gamma_2<\gamma_4$. Therefore $0\le\eta_1\cdot\gamma_5+\eta_2\le r_{2,\min}\le r_2(x)\le r_{2,\max}\le(\eta_1\cdot\gamma_3+\eta_2)+\sqrt{\eta_1^2+\eta_3^2}\cdot\tilde{\gamma}$. One can verify that $\left.\frac{dr_2}{dx}\right|_{x=\gamma_4}=2\eta_1<0$.

We also have $r_1+r_2=2(\eta_1\cdot x+\eta_2)\Rightarrow\frac{dr_1}{dx}+\frac{dr_2}{dx}=2\eta_1\Rightarrow\left.\frac{dr_1}{dx}\right|_{x=\gamma_4}=-\left.\frac{dr_2}{dx}\right|_{x=\gamma_4}+2\eta_1=0$ and $\frac{d^2r_1}{dx^2}=-\frac{d^2r_2}{dx^2}>0,\forall x\in[\gamma_1,\gamma_5]$. Then r_1 reaches a minimum value of $(\eta_1\cdot\gamma_3+\eta_2)-\sqrt{\eta_1^2+\eta_3^2}\cdot\tilde{\gamma}$ at $x=\gamma_4$. The other root of the equation $0=\eta_1^2\cdot(x-\gamma_1)\cdot(\gamma_5-x)-\eta_3^2\cdot(2\gamma_3-x)^2$ is $x=\gamma_2$. One can verify that $\left.\frac{dr_1}{dx}\right|_{x=\gamma_2}=2\eta_1$. In equation (3.2), $\eta_1<0$, so $\left.\frac{dr_1}{dx}\right|_{x=\gamma_2}<0$. The condition $\eta_2\ge\eta_3\cdot\tilde{\gamma}-\eta_1\cdot\gamma_1$ implies $(\eta_1\cdot\gamma_3+\eta_2)-\sqrt{\eta_1^2+\eta_3^2}\cdot\tilde{\gamma}\ge(\eta_1+\eta_3-\sqrt{\eta_1^2+\eta_3^2})\cdot\tilde{\gamma}\ge 0$ so

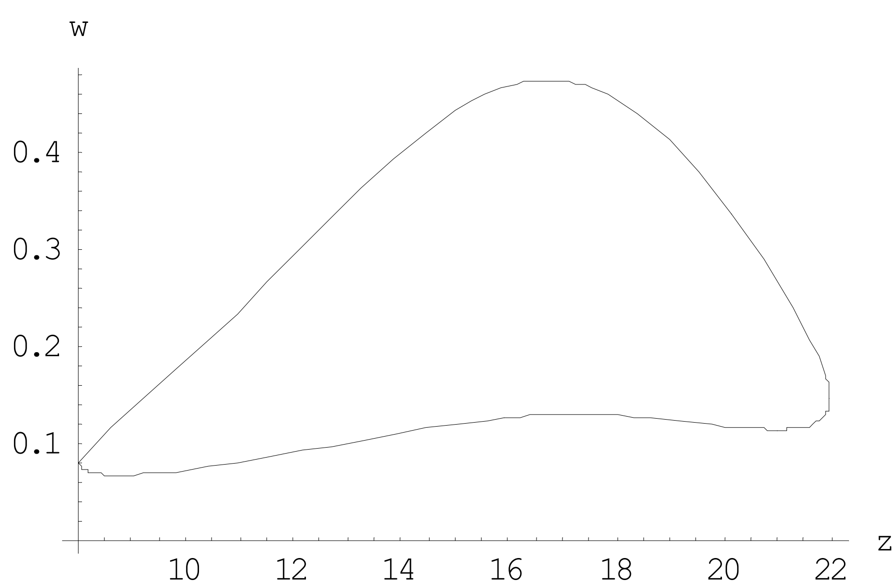
$0\le(\eta_1\cdot\gamma_3+\eta_2)-\sqrt{\eta_1^2+\eta_3^2}\cdot\tilde{\gamma}\le r_{1,\min}\le r_1(x)\le r_{1,\max}\le\eta_1\cdot\gamma_1+\eta_2$.

If $\eta_1 = 0$, then $\eta_3^2 \cdot (x-\gamma_3)^2 + (y-\eta_2)^2 = \eta_3^2 \cdot \tilde{\gamma}^2$. One can easily verify that $\gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < \gamma_5$. Verify with the graph of Shifted Ellipse 4.3 that $r_{2,\max}$ occurs to the left of $r_{1,\min}$.

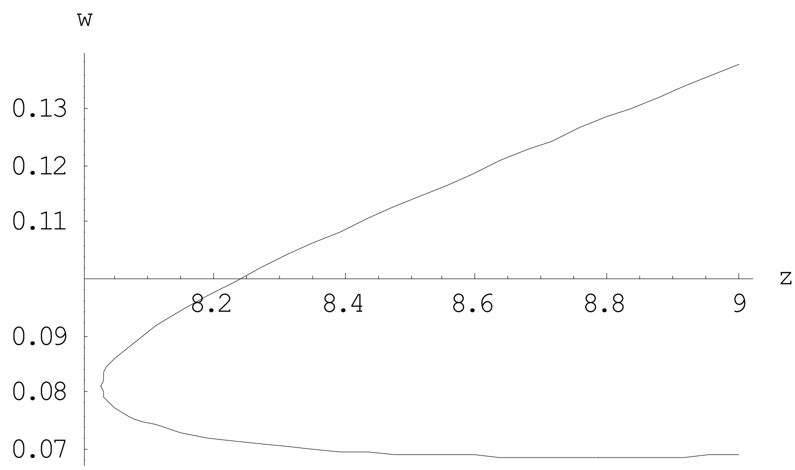
We may now piece together bounds on linear combinations of the α-th powers over constants. Clearly, one can put only rational powers for α into the computer for plotting. In Section 7 we will choose $c_1, c_2 > 0$ and $\alpha = -5/2 < 0$. Then $w_{\vec{c},\min} \geq c_1 \cdot r_{1,\max}^{\alpha} + c_2 \cdot r_{2,\max}^{\alpha}$ and $w_{\vec{c},\max} \leq c_1 \cdot r_{1,\min}^{\alpha} + c_2 \cdot r_{2,\min}^{\alpha}$ which implies $w_{\vec{c},\min} \geq c_1 \cdot (\eta_1 \cdot \gamma_5 + \eta_2)^{\alpha} + c_2 \cdot (\eta_1 \cdot \gamma_3 + \eta_2 + \sqrt{\eta_1^2 + \eta_3^2} \cdot \tilde{\gamma})^{\alpha}$ and $w_{\vec{c},\max} \leq c_1 \cdot (\eta_1 \cdot \gamma_3 + \eta_2 - \sqrt{\eta_1^2 + \eta_3^2} \cdot \tilde{\gamma})^{\alpha} + c_2 \cdot (\eta_1 \cdot \gamma_1 + \eta_2)^{\alpha}$. Hence, it is sufficient to bound the plot of $w_{\vec{c}}$ over the interval $\gamma_1 \leq x \leq \gamma_5$ by this range.

Section 7. The α-ellipse

Graph 7.1 of the α-ellipse. We might as well choose one of the constants, c_1 or c_2, in Definitions 1.11 and 1.12 to be 1 since the scale of $w_{\vec{c}} = c_1 \cdot r_1^{\alpha} + c_2 \cdot r_2^{\alpha}$ is arbitrary. We choose $c_1 = 5$, $c_2 = 1$, and $\alpha = -5/2$ so $w_{top} = 5 \cdot r_1^{-2.5} + 1 \cdot r_2^{-2.5}$ and $w_{bottom} = 1 \cdot r_1^{-2.5} + 5 \cdot r_2^{-2.5}$. Then the real-valued locus of w_{top} and w_{bottom} is a closed smooth curve without cusps which looks like



Graph 7.2. A closer examination of the left critical point of Graph 7.1 reveals that this α -ellipse is indeed smooth at $x = \gamma_1 = 15 - \frac{7}{221}\sqrt{48449} <\approx 8.02815$.



Therefore $\eta_1 = -\frac{B}{2C} = -\frac{73920}{2 \cdot 436041} \approx -0.0847627$, $\eta_2 = -\frac{E}{2C} = \frac{5469210}{2 \cdot 436041} \approx 6.27144$,

$\eta_3 = \frac{\sqrt{\pi_2}}{2C} \approx 0.336030$, $\gamma_2 \approx 13.2948$, $\gamma_3 = \frac{2CD - BE}{B^2 - 4AC} = 15$, $\gamma_4 \approx 16.7052$,

$\gamma_5 = 15 + \frac{7}{221}\sqrt{48449} >\approx 21.97185$, $\tilde{\gamma} = \frac{\sqrt{\pi_1^2 + \pi_0 \pi_2}}{\pi_2} = \frac{7}{221}\sqrt{48449} \approx 6.97185$

$r_{2,\max} = r_2(\gamma_2) = \eta_1 \cdot \gamma_3 + \eta_2 + \sqrt{\eta_1^2 + \eta_3^2} \cdot \tilde{\gamma} \approx 7.41614$, $r_{1,\max} = r_1(\gamma_1) = \eta_1 \cdot \gamma_1 + \eta_2 \approx 5.59095$

$r_{2,\min} = r_2(\gamma_5) = \eta_1 \cdot \gamma_5 + \eta_2 \approx 4.40905$, $r_{1,\min} = r_1(\gamma_4) = \eta_1 \cdot \gamma_3 + \eta_2 - \sqrt{\eta_1^2 + \eta_3^2} \cdot \tilde{\gamma} \approx 2.58386$

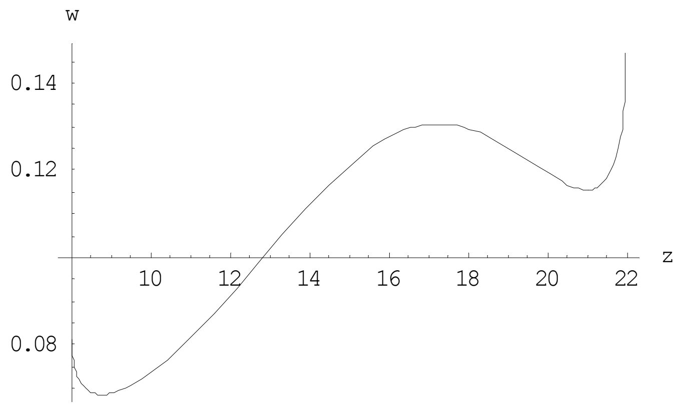
where the symbol $<\approx$ means "approximately but less than". Observe that

$\eta_2 = 6.27144 \geq 3.02324 = \eta_3 \cdot \tilde{\gamma} - \eta_1 \cdot \gamma_1$.

$w_{top,\min,est} = 0.0743247 \approx 5 \cdot r_{1,\max}^{\alpha} + 1 \cdot r_{2,\max}^{\alpha} \leq w_{top} \leq 5 \cdot r_{1,\min}^{\alpha} + 1 \cdot r_{2,\min}^{\alpha} \approx 0.490403 = w_{top,\max,est}$

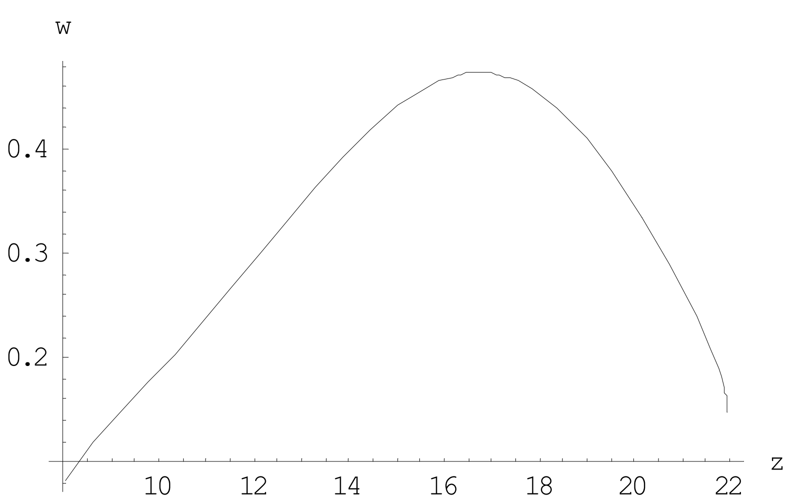
$w_{bottom,\min,est} = 0.0469126 \approx 1 \cdot r_{1,\max}^{\alpha} + 5 \cdot r_{2,\max}^{\alpha} \leq w_{bottom} \leq 1 \cdot r_{1,\min}^{\alpha} + 5 \cdot r_{2,\min}^{\alpha} \approx 0.215673 = w_{bottom,\max,est}$

Graph 7.3. Approximation of the critical points of the bottom. A closer inspection of the plot of w_{bottom}



and repeated computation of the derivative $\frac{dw_{bottom}}{dx}$ with Mathematica near the critical points reveals that $w_{bottom,\max} \approx 0.146991$ at $x = \gamma_5$ and $w_{bottom,\min} \approx 0.0686117$ at $x \approx 8.77599 < \gamma_2$. Therefore, the estimated bounds $w_{bottom,\min,est}$ and $w_{bottom,\max,est}$ are within range, i.e. $w_{bottom,\min,est} \le w_{bottom,\min} < w_{bottom,\max} \le w_{bottom,\max,est}$, with errors of $\frac{w_{bottom,\min,est} - w_{bottom,\min}}{w_{bottom,\min}} \approx -31.6259\%$ and $\frac{w_{bottom,\max,est} - w_{bottom,\max}}{w_{bottom,\max}} \approx 46.7253\%$. The most important feature about w_{bottom} is that it has a local minimum $(17.2629, 0.130664)$ and a local maximum $(20.9929, 0.115512)$, both with $\gamma_4 < x < \gamma_5$, neither of which is global.

Graph 7.4. Approximation of the critical points of the top. A closer inspection of the plot of w_{top} in Graph 7.4



and repeated computation of the derivative $\frac{dw_{top}}{dx}$ with Mathematica near the critical points reveals that $w_{top,\max} \approx 0.473283$ at

$x \approx 16.723731 > \gamma_4$ and $w_{top,\min} \approx 0.0811777$ at $x = \gamma_1$. Therefore, the estimated bounds $w_{top,\min,est}$ and $w_{top,\max,est}$ are within range, i.e. $w_{top,\min,est} \le w_{top,\min} < w_{top,\max} \le w_{top,\max,est}$, with errors of

$\dfrac{w_{top,\min,est} - w_{top,\min}}{w_{top,\min}} \approx -8.44074\%$ and $\dfrac{w_{top,\max,est} - w_{top,\max}}{w_{top,\max}} \approx 3.61729\%$.

Section 8. Differential resolvents are useful.

It is natural to try to solve the equation $\dfrac{1}{\alpha}\dfrac{dw_{\vec{c}}}{dz} = c_1 \cdot r_1^{\alpha-1} \cdot \dfrac{dr_1}{dz} + c_2 \cdot r_2^{\alpha-1} \cdot \dfrac{dr_2}{dz} = 0$ for z and determine the number of such real solutions z which fall in the interval $[\gamma_1, \gamma_5]$. But computing exact maxima and minima of $w_{\vec{c}}$ even for integral α by straight algebraic elimination is too complicated to present here, and it fails to work for $\alpha \notin \mathbb{Q}$. Plotting $w_{\vec{c}}$ versus x over the interval $[\gamma_1, \gamma_5]$ will reveal the existence of extreme points. But it does not mathematically rule out the existence of other extrema which fail to be revealed due to insufficient resolution of the plot. However, in the case of rational α, we are at least guaranteed the following theoretical possibility of finding a minimal polynomial for $\dfrac{dw}{dz}$ over $\mathbb{Q}(z)$. The z-roots of the term, which is lowest in order of $\dfrac{dw}{dz}$, gives all the possible locations of the critical points. We do not intend to pursue this method, but we do wish to demonstrate the size of the calculation of a minimal polynomial for arbitrary linear combinations over constants of powers of the roots. So, in Section 9 we will show how one would go about computing the minimal polynomial for w over $\mathbb{Q}(z)$ in the case $\alpha = -\dfrac{5}{2}$.

Lemma 8.1. When $\alpha, c_1, c_2 \in \mathbb{Q}$, both the z and w coordinates of the critical points of $w = c_1 \cdot r_1^{\alpha} + c_2 \cdot r_2^{\alpha}$ are algebraic numbers.

Proof. Since $\frac{dr_1}{dz}$ and $\frac{dr_2}{dz}$ are algebraic over $\mathbb{Q}(z)$ (one can use (2.7) to express them in terms of r, Δ and W), it follows that the z-roots of $\frac{dw}{dz} = \alpha \cdot \{c_1 \cdot r_1^{\alpha-1} \frac{dr_1}{dz} + c_2 \cdot r_2^{\alpha-1} \frac{dr_2}{dz}\}$ are algebraic numbers. Thus, the w coordinate is an algebraic number. □

Using Sturm Sequences to determine upper bounds on the number of critical points of the α-ellipse

Lemma 8.2. Let $R_2(x)$, $R_1(x)$, and $R_0(x)$ be arbitrary differentiable functions which do not change sign or vanish on the open interval I and let $R_2(x) \cdot R_0(x) > 0$ on I. Let $w(x)$ be a *particular* solution of $R_2(x) \cdot w'' + R_1(x) \cdot w' + R_0(x) \cdot w = 0$ such that $w(x) > 0, \forall x \in I$. Then w' cannot go from $-$ to $+$ anywhere on I.

Proof. Let $x_0 \in I$ be a point where w' crosses from $-$ to $+$. Then on some open neighborhood U of x_0 contained in I, $w''(x) > 0$. Hence $R_2(x) \cdot (+) + R_1(x) \cdot (-) + R_0(x) \cdot (+) = 0$ goes to $R_2(x) \cdot (+) + R_1(x) \cdot (+) + R_0(x) \cdot (+) = 0$ on U. But the condition $R_2(x) \cdot R_0(x) > 0$ would imply $R_1(x_0) = 0$, a contradiction with the hypothesis that $R_1(x)$ does not vanish on I. □

Theorem 8.3. Let $R_2(x)$, $R_1(x)$, and $R_0(x)$ be arbitrary differentiable functions of the real variable x which do not change sign or vanish on an open interval I. Let $w(x)$ be a *particular* solution of $R_2(x) \cdot w'' + R_1(x) \cdot w' + R_0(x) \cdot w = 0$ such that $w(x) > 0, \forall x \in I$. Then w' can change sign at most once on I.

Proof. First, let us rule out the possibility that $\exists x' \in I \ni w''(x') = 0 = w'(x')$. Since $R_2(x)$, $R_1(x)$, and $R_0(x)$ are differentiable on I, they are continuous on I. Therefore they have no singularities on I. Therefore $w' = 0 \Rightarrow R_1 \cdot w' = 0$ and $w'' = 0 \Rightarrow R_2 \cdot w'' = 0$. Therefore $w''(x') = 0 = w'(x') \Rightarrow R_0(x') \cdot w(x') = 0$. By hypothesis, neither $R_0(x)$ nor $w(x)$ vanish on I, a contradiction.

Let $x_0 \in I$ be a point where w' crosses from $-$ to $+$. Then on some real open neighborhood U of x_0 contained in I, $w''(x) > 0$. Hence $R_2(x)\cdot(+) + R_1(x)\cdot(-) + R_0(x)\cdot(+) = 0$ goes to $R_2(x)\cdot(+) + R_1(x)\cdot(+) + R_0(x)\cdot(+) = 0$ on U. If w' goes from $+$ to $-$ at another point $x_1 \in I$, then in some open neighborhood V of x_0 contained in I, $w''(x) < 0$. Then $R_2(x)\cdot(-) + R_1(x)\cdot(+) + R_0(x)\cdot(+) = 0$ goes to $R_2(x)\cdot(-) + R_1(x)\cdot(-) + R_0(x)\cdot(+) = 0$ on V.

If $R_2(x)\cdot R_1(x) > 0$ on I, then $R_2(x)\cdot(+) + R_1(x)\cdot(+) + R_0(x)\cdot(+) = 0$ on U and $R_2(x)\cdot(-) + R_1(x)\cdot(-) + R_0(x)\cdot(+) = 0$ on V would contradict the hypothesis that $R_0(x)$ does not change sign on I, since they would imply $R_0(x)$ would have different signs on U and V.

Similarly, if $R_1(x)\cdot R_0(x) > 0$ on I, then $R_2(x)\cdot(+) + R_1(x)\cdot(+) + R_0(x)\cdot(+) = 0$ on U and $R_2(x)\cdot(-) + R_1(x)\cdot(+) + R_0(x)\cdot(+) = 0$ on V would contradict the hypothesis that $R_2(x)$ does not change sign on I, since they would imply $R_2(x)$ would have different signs on U and V.

If $R_2(x)\cdot R_1(x) < 0$ and $R_1(x)\cdot R_0(x) < 0$ on I, then $R_2(x)\cdot R_0(x) > 0$ on I. But then w' could not go from $-$ to $+$ by Lemma 8.2, a contradiction with the hypothesis that such a point $x_0 \in I$ where w' crosses from $-$ to $+$ exists. □

Next, in Theorem 8.4, we specialize $R_2(x)$, $R_1(x)$, and $R_0(x)$ to polynomials parametrized by the real variable α. Therefore, for each $\alpha \in \mathbb{R}$, there will exist finitely many $x \in [\gamma_1, \gamma_5]$ (the x-zeroes) which make $R_2(x)$, $R_1(x,\alpha)$, and $R_0(x,\alpha)$ vanish, with the exception of $\alpha = 0$ for which $R_0(x,\alpha) \equiv 0, \forall x \in \mathbb{R}$.

Theorem 8.4. Let $R_0(x,\alpha), R_1(x,\alpha) \in \mathbb{R}[x,\alpha]$ be the polynomials in x and α given by Definition 2.4 as the terms of α-resolvent (2.3) of polynomial (3.2) with coefficient-functions given by (3.3). Then $R_0(x,\alpha)$ has at most $K_{0,\alpha}$ x-zeroes and $R_1(x,\alpha)$ has at most $K_{1,\alpha}$ x-zeroes on $[\gamma_1, \gamma_5]$, where $K_{0,\alpha}$ and $K_{1,\alpha}$ are given in Table 8.5 for the appropriate interval of α.

Proof. The standard Sturm sequence of a polynomial is defined on page 313 of [2]. Let $\{S_{10}(\alpha,x),S_{11}(\alpha,x),S_{12}(\alpha,x),S_{13}(\alpha,x),S_{14}(\alpha,x)\}$ denote the standard Sturm sequence of $R_1(x,\alpha)$, and let $\{S_{00}(\alpha,x),S_{01}(\alpha,x),S_{02}(\alpha,x),S_{03}(\alpha,x)\}$ denote the standard Sturm sequence of $R_0(x,\alpha)$. So $S_{10}(\alpha,x)\equiv R_1(x,\alpha)$, $S_{ik}(\alpha,x)\in\mathbb{Q}(\alpha)[x]$, $\deg_x S_{1k}(\alpha,x)=4-k$, $\deg_x S_{0k}(\alpha,x)=3-k$. In the Mathematica program Convex.nb, we denote $S_{0k}(\alpha,x)$ by R0Sturmk and $S_{1k}(\alpha,x)$ by R1Sturmk. We then compute $S_{1k}(\alpha,\gamma_1)$ and $S_{1k}(\alpha,\gamma_5)$ for $k\in[4]_0$ and $S_{0k}(\alpha,\gamma_1)$, $S_{0k}(\alpha,\gamma_5)$ for $k\in[3]_0$, which we denote in Convex.nb by R1Sturmk1, R1Sturmk5, R0Sturmk1, R0Sturmk5, respectively.

We then computed and collected the real α -zeroes of all the numerators and denominators of $\{S_{1k}(\alpha,\gamma_1)\}_{k=0}^4$, $\{S_{1k}(\alpha,\gamma_5)\}_{k=0}^4$, $\{S_{0k}(\alpha,\gamma_1)\}_{k=0}^3$, $\{S_{0k}(\alpha,\gamma_5)\}_{k=0}^3$, which we denote in Convex.nb by NR1Sturmk1, NR1Sturmk5, NR0Sturmk1, NR0Sturmk5, DR1Sturmk1, DR1Sturmk5, DR0Sturmk1, DR0Sturmk5. We list these α -zeroes in order in (8.1). For any fixed $x\in[\gamma_1,\gamma_5]$, the sign of any member of these four sets of Sturm sequences remains constant for all α between any consecutive members in list (8.1).

$$\begin{gathered}-36.5368 < -18.8792 < -15.1068 < -8.4845 < -7.35858 < -6.73418 < -4.51518 < \\ -3.52892 < -1.46538 < -0.66784 < 0 < 0.0834687 < 0.512623 < 0.525273 < \\ 0.956701 < 0.956825 < 0.972454 < 0.989995 < 0.996921 < 0.997527 < \\ 0.998094 < 0.998269 < 1 < 1.00062 < 1.00478 < 1.11208 < 1.27406 < \\ 1.29178 < 1.59394 < 1.60198 < 3.52892 < 9.73316 < 45.2156 < 221.306 \qquad (8.1)\end{gathered}$$

We then computed the number of sign changes in the sequences of five numbers $\{S_{1k}(\alpha,\gamma_1)\}_{k=0}^4$ and $\{S_{1k}(\alpha,\gamma_5)\}_{k=0}^4$ and the sequences of four numbers $\{S_{0k}(\alpha,\gamma_1)\}_{k=0}^3$ and $\{S_{0k}(\alpha,\gamma_5)\}_{k=0}^3$ for a random value of α between each consecutive pair of α -zeroes in list (8.1).

The column in Table 8.5 headed by $\mathrm{K}_{1,\alpha}$ is the maximum number of sign changes possible for $R_1(x,\alpha)$ on the interval $[\gamma_1,\gamma_5]$. The column headed by $\mathrm{K}_{0,\alpha}$ is the maximum number of sign changes possible for $R_0(x,\alpha)$ on the interval $[\gamma_1,\gamma_5]$.

Table 8.5

	$K_{1,\alpha}$	$K_{0,\alpha}$	$\geq \#x \in [\gamma_1, \gamma_5] \ni w' = 0$
$-\infty < \alpha < -36.5368$	4	3	16
$-36.5368 < \alpha < -18.8792$	4	1	14
$-18.8792 < \alpha < -3.52892$	2	1	12
$-3.52892 < \alpha < 0$	0	1	10
$\alpha = 0$	0	0	9
$0 < \alpha < 0.525273$	0	1	10
$0.525273 < \alpha < 1$	0	0	9
$\alpha = 1$	1	1	11
$1 < \alpha < 1.60198$	0	0	9
$1.60198 < \alpha < 45.2156$	0	1	10
$45.2156 < \alpha < 221.306$	2	1	12
$221.306 < \alpha < \infty$	2	3	14

□

Observe that $K_{1,\alpha} \leq 4 = \deg_x R_1(x,\alpha)$ and $K_{0,\alpha} \leq 3 = \deg_x R_0(x,\alpha)$ for all α.

Corollary 8.6. The number of real-valued critical points of any particular solution w of α-resolvent (2.3), with coefficient-functions given by (3.3), with $w(x) > 0, \forall x \in [\gamma_1, \gamma_5]$, is bounded above by the integer given in the column headed by $\geq \# x \in [\gamma_1, \gamma_5] \ni w' = 0$ of Table 8.5 for the given range on α.

Proof. We must now enumerate all the cases not covered by Theorem 8.3 where w' could vanish. Therefore, for each $\alpha \in \mathbb{R}^{\#}$ we must determine if $w' = 0$ when $R_2(x) = 0$ or $R_1(x,\alpha) = 0$ or $R_0(x,\alpha) = 0$. The case $\alpha = 0 \Rightarrow w' \equiv 0, \forall x$ is exceptional.

The formula following Definition 3.1 shows that W is linear in x. Therefore, W, and therefore $R_2(x)$ by (2.4), changes sign exactly *once* at some point $x_2 \in [\gamma_1, \gamma_5]$. Since $R_2(x)$ is independent of α by (2.4), x_2 is independent of α. It is easy to see from Definitions 1.12, 2.1, & 3.2 that $w'(\gamma_1) = w'(\gamma_5) = \infty \neq 0$. If $w'(x_2) = 0$ and $R_2(x_2) = 0$, then the resolvent and $w(x_2) > 0$ imply $R_0(x_2, \alpha) = 0$, which is quadratic in α. But Mathematica reveals the resultant of $R_0(x,\alpha)$ and $R_2(x)$, eliminating x, to be 1, independent of α ! So $R_0(x_2,\alpha)=0$ has no real α -root! Hence, for $\forall\alpha\in\mathbb{R}^{\#}$, $w'(x_2)\neq 0$.

For any α we have at most *four* real values of x_1 in $[\gamma_1,\gamma_5]$ such that $R_1(x_1,\alpha)=0$ and $w'(x_1)=0$, where x_1 depends on α, since $\deg_x R_1(x,\alpha)=4$, and at most *three* real values of x_0 in $[\gamma_1,\gamma_5]$ such that $w'(x_0)=0$ and $R_0(x_0,\alpha)=0$, where x_0 depends on α. Therefore, we have a total of *seven* possible critical points in addition to the number of critical points, C, at which $R_2(x)\cdot R_1(x,\alpha)\cdot R_0(x,\alpha)\neq 0$. By Theorem 8.3, C equals at most the number of subintervals of $[\gamma_1,\gamma_5]$ where $R_2(x)\cdot R_1(x,\alpha)\cdot R_0(x,\alpha)\neq 0$. Therefore, by Theorem 8.4, $C=\mathrm{K}_{1,\alpha}+\mathrm{K}_{0,\alpha}+2$ in Table 8.5, where the 2 accounts for the additional point where $R_2(x)=0$ and the fact that the number of subintervals is 1 more than the number of points which break them up. So, the last column is $C+7=\mathrm{K}_{1,\alpha}+\mathrm{K}_{0,\alpha}+9$. □

A more detailed analysis could probably rule out many of the possible critical points of $w(x)$ in Corollary 8.6. But, for now, we assert only the weakest, but most rigorous, result.

Section 9. Computing the minimal polynomial of w.

We may compute the minimal polynomial of $w_{top}=5\cdot r_1^{-5/2}+1\cdot r_2^{-5/2}$ and $w_{bottom}=1\cdot r_1^{-5/2}+5\cdot r_2^{-5/2}$ over $\mathbb{Z}[A,B,C,D,E,F,z]=\mathbb{Z}[z]$ by setting up the following system of equations for w_{top}

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}=\begin{bmatrix}5 & 1 & 0 & -w_{top}^1\\ 0 & 0 & 10 & (25\cdot r_1^{-5}+r_2^{-5})-w_{top}^2\\ (125r_1^{-5}+15r_2^{-5}) & (75r_1^{-5}+r_2^{-5}) & 0 & -w_{top}^3\\ 0 & 0 & 5\cdot(75r_1^{-5}+r_2^{-5}) & (625r_1^{-10}+150r_1^{-5}r_2^{-5}+r_2^{-10})-w_{top}^4\end{bmatrix}\cdot\begin{bmatrix}r_1^{-5/2}\\ r_2^{-5/2}\\ (r_1\cdot r_2)^{-5/2}\\ 1\end{bmatrix}$$

We then eliminate $r_1^{-5/2}$ and $r_2^{-5/2}$ by taking the determinant of the 4×4 matrix on the right to get $(625\cdot r_2^{10}-200\cdot r_1^5 r_2^5-r_1^{10})-(75\cdot r_1^5 r_2^{10}+r_1^{10}r_2^5)\cdot w^2+r_1^{10}r_2^{10}\cdot w^4=0$. We may then use resultants to eliminate r_1 and r_2 between this equation and equations (3.2) $P(r_1)=0$ and $P(r_2)=0$.

The resulting polynomial has the form

$$P_{\bar{c}}(w) = M(z,w) = P_0 + P_2 \cdot w^2 + P_4 \cdot w^4 + P_6 \cdot w^6 + P_8 \cdot w^8 + P_{10} \cdot w^{10} + P_{12} \cdot w^{12} + P_{14} \cdot w^{14} + P_{16} \cdot w^{16}$$

with $P_{2m} \in \mathbb{Z}[z]$, $\deg_x P_{2m} = 40 + 5m$, and integer coefficients with as many as 275 digits.

Section 10. Convex functions

Definition 10.1. A *convex* function $g(x)$ on the real closed interval $[a,b]$ is a real-valued function of a real variable x such that $g(\theta \cdot x' + (1-\theta) \cdot x'') \le \theta \cdot g(x') + (1-\theta) \cdot g(x'') \forall x', x'' \in [a,b], \theta \in [0,1]$.

Conjecture 10.2. The author's experimental discovery that the number of local maxima and minima of $w = c_1 \cdot r_1^\alpha + c_2 \cdot r_2^\alpha$ for various values of $c_1, c_2, \alpha \in \mathbb{R}$ seems to be bounded above by 3 led to the question of whether this bound follows from some basic fact about the difference of convex functions. This led to the conjecture: if $g(x)$ and $h(x)$ are real-valued convex functions on an interval $a \le x \le b$ which are not constant on any subinterval, can $g(x) - h(x)$ cross a value ϖ an arbitrary number of times? The following example demonstrates that $g(x) - h(x)$ can change signs an arbitrary but finite number of times on a finite interval $-1 \le x \le 1$ even when $g(x)$ and $h(x)$ are both *strictly* convex.

Counterexample 10.3. Take the regular N-gon with corners at the coordinates $u_k \equiv (\cos(\frac{2\pi}{N}k), \sin(\frac{2\pi}{N}k))$ for $1 \le k \le N$ and the rotated N-gon with corners at the coordinates $v_k \equiv (\cos(\frac{(2k+1)\pi}{N}), \sin(\frac{(2k+1)\pi}{N}))$. The bottom halves of these N-gons are convex functions which cross roughly $N/2$ times. To prove this algebraically, first observe that by symmetry we need to prove only that one edge of one of the N-gons crosses one of the edges of the other N-gon. This reduces in the case of proving that the vertical line segment connecting $v_{N-1} = (\cos(\frac{\pi}{N}), -\sin(\frac{\pi}{N}))$ to $v_N = (\cos(\frac{\pi}{N}), \sin(\frac{\pi}{N}))$ intersects the line segment connecting $u_N \equiv (1,0)$ to $u_1 \equiv (\cos(\frac{2\pi}{N}), \sin(\frac{2\pi}{N}))$.

Let $\theta \equiv \frac{\pi}{N}$. The equation for the line segment connecting u_N and u_1 is $y = \frac{\sin(2\theta) - 0}{\cos(2\theta) - 1} \cdot (x-1)$ over $\cos(2\theta) \le x \le 1$. We need to prove that $y\big|_{x=\cos\theta} < \sin(\theta)$. But

$$\begin{aligned} y\big|_{x=\cos\theta} &= \frac{\sin(2\theta)}{\cos(2\theta)-1}(\cos\theta - 1) = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta - 1}(\cos\theta - 1) \\ &= \frac{2\sin\theta\cos\theta}{-2\sin^2\theta}(\cos\theta - 1) = \frac{\cos\theta - \cos^2\theta}{\sin\theta} = \frac{\sqrt{1-\sin^2\theta} - 1 + \sin^2\theta}{\sin\theta} \end{aligned}$$

. Let $\upsilon \equiv \sin\theta$. Then

$y\big|_{x=\cos\theta} = \upsilon + \frac{\sqrt{1-\upsilon^2}}{\upsilon} - \frac{1}{\upsilon}$. For $0 < \upsilon < 1$, we have $y\big|_{x=\cos\theta} = \upsilon + \frac{\sqrt{1-\upsilon^2}}{\upsilon} - \frac{1}{\upsilon} < \upsilon$. □

However, even in the special case of equation (3.2), proving the convexity or concavity of just r_1^α over part of the real domain $[\gamma_1, \gamma_5]$ proved to be quite a challenge, as one can see by the computer calculations necessary in the proof of Theorem 10.11.

Lemma 10.4. (Theorem 7.37 in [10]) Let $\phi(x)$ be real-valued and twice differentiable on $a < x < b$. Then $\phi(x)$ is monotone increasing iff $\phi'(x) \ge 0$ on $a < x < b$ and $\phi'(x)$ is monotone increasing iff $\phi''(x) \ge 0$ on $a < x < b$.

Lemma 10.5. If $\phi(x)$ and $\psi(x)$ are both monotone increasing (resp. decreasing) and of the same sign $\phi(x) \cdot \psi(x) \ge 0$ on $a < x < b$, then their product $\phi(x) \cdot \psi(x)$ is monotone increasing (resp. decreasing) if $\phi(x) \ge 0$ and monotone decreasing (resp. increasing) if $\phi(x) \le 0$.

Proof. The sign of the derivative $(\phi \cdot \psi)' = \phi' \cdot \psi + \phi \cdot \psi'$ equals $\text{sgn}(\phi') \cdot \text{sgn}(\phi) = \text{sgn}(\psi') \cdot \text{sgn}(\psi)$. The result follows from Lemma 10.4. □

Theorem 10.6. Let $\phi(x)$ be real-valued, twice differentiable, and positive on the interval $a < x < b$. Let $\alpha \in \mathbb{R}$ with $\alpha \ge 1$.

Case 10.6a. If $\phi'(x)$ is monotone increasing and $\phi(x)$ is either monotone increasing or monotone decreasing then ϕ^α is convex.

Case 10.6b. If $\phi'(x)$ is monotone decreasing and $\phi(x)$ is either monotone decreasing or monotone increasing then $-\phi^\alpha$ is convex.

Proof.

Case 10.6a.1. $\phi(x)$ is monotone increasing, so $\phi' > 0$. Each of the factors on the right-hand-side of $(\phi^{\alpha})' = \alpha \cdot \phi^{\alpha-1} \cdot \phi'$ is positive and monotone increasing, therefore their product $(\phi^{\alpha})'$ is monotone increasing. By Lemma 10.4 ϕ^{α} is convex.

Case 10.6a.2. $\phi(x)$ is monotone decreasing, so $\phi' < 0$. Each of the factors $-\phi^{\alpha-1}$ and ϕ' on the right-hand-side of $(\phi^{\alpha})' = (-\alpha) \cdot (-\phi^{\alpha-1}) \cdot \phi'$ is monotone increasing and negative, therefore their product $(-\phi^{\alpha-1}) \cdot \phi'$ is monotone decreasing by Lemma 10.5. Since $-\alpha < 0$, $(\phi^{\alpha})'$ is monotone increasing.

Case 10.6b.1. $\phi(x)$ is monotone increasing, so $\phi' > 0$. Each of the factors on the right-hand-side of $(-\phi^{\alpha})' = \alpha \cdot \phi^{\alpha-1} \cdot (-\phi')$ is monotone increasing and positive, therefore their product is monotone increasing.

Case 10.6b.2. $\phi(x)$ is monotone decreasing, so $\phi' < 0$. Each of the factors on the right-hand-side of $(-\phi^{\alpha})' = \alpha \cdot \phi^{\alpha-1} \cdot (-\phi')$ is monotone increasing and positive, therefore their product is monotone increasing. □

Corollary 10.7. Any solution w of the α -resolvent (2.3) and (3.3) is the difference of two convex functions when $\alpha \geq 1$.

Proof. By equation (3.2), r_1 fits Case 10.6a and r_2 fits Case 10.6b. Therefore $w = c_1 \cdot r_1^{\alpha} - c_2 \cdot (-r_2^{\alpha})$ is the difference of two convex functions when $\alpha \geq 1$. □

Theorem 10.8. Let $\phi(x)$ be real-valued, twice differentiable and positive with $\phi''(x) \leq 0$ on the interval $a < x < b$. Let $\alpha \in \mathbb{R}$. Then $-\phi^{\alpha}$ is convex on $a < x < b$ when $0 < \alpha < 1$ and ϕ^{α} is convex when $\alpha < 0$.

Proof. The second derivative $(\phi^{\alpha})'' = \alpha \cdot \{(\alpha - 1) \cdot (\phi')^2 + \phi \cdot \phi''\} \cdot \phi^{\alpha-2}$ is easily seen to be nonnegative on $a < x < b$ when $\alpha < 0$ and nonpositive when $0 < \alpha < 1$. By Lemma 10.4 $-\phi^{\alpha}$ is convex when $0 < \alpha < 1$, and ϕ^{α} is convex when $\alpha < 0$. □

Corollary 10.9. For equation (3.2) r_2 fits theorem 10.8. Therefore $-c_2 \cdot r_2^\alpha$ is convex when $0<\alpha<1$ and $c_2 \cdot r_2^\alpha$ is convex when $\alpha<0$.

The following example demonstrates that we do not have as easy a result when $\phi''(x) \ge 0$.

Example 10.10. Let $\phi(x)=x^2$ with $x>0$. Then $\phi''(x)\ge 0$, and ϕ^α is convex on $x>0$ when $\frac{1}{2} \le \alpha$ and $\alpha<0$ and concave on $x>0$ when $0<\alpha<\frac{1}{2}$.

The 1999 Dell computer running on Windows 98 with Pentium 2 processor could not prove a result on the general quadratic $r^2 - e_1 \cdot r + e_2 = 0$ with indeterminate e_1, e_2 or even on the bi-quadratic (3.1) with indeterminate A,B,C,D,E,F. However, we were able to prove a result for equation (3.2).

Theorem 10.11. Let r_1 be the "lower" root of equation (3.2) on the interval $\gamma_1 \le x \le \gamma_5$. Let $\alpha \in \mathbb{R}$. Then *it is mostly true* that for $|\alpha| \ge 6$ r_1^α is concave for $x \in [\gamma_1, 9]$, convex for $x \in [9,11]$, concave for $x \in [11,20]$, and convex for $x \in [20, \gamma_5]$. For $|\alpha|<6$, r_1^α is concave for $\forall x \in [\gamma_1, \gamma_5]$.

Proof. *Observe that the proof of Theorem 10.11 does not require the full* α *-resolvent (2.3) with terms (3.3) in contrast to Theorem 8.4*. The inhomogeneous first-power resolvent (2.7) of $r^2 - e_1 \cdot r + e_2 = 0$ in integral form is $2\Delta \cdot r' = \Delta' \cdot r + 2W$ where $W = e_1 \cdot e_2' - 2e_1' \cdot e_2$. Differentiation leads to $4\Delta^2 \cdot r'' = (2\Delta \cdot \Delta'' - (\Delta')^2) \cdot r + 2(2\Delta \cdot W' - \Delta' \cdot W)$. Therefore

$$
\begin{aligned}
\frac{4\Delta^2 \cdot (r^\alpha)''}{\alpha \cdot r^{\alpha-2}} &= (\alpha-1)\cdot(2\Delta \cdot r')^2 + r \cdot (4\Delta^2 r'') \\
&= (\alpha-1)\cdot(\Delta' \cdot r + 2W)^2 + r \cdot ((2\Delta \cdot \Delta'' - (\Delta')^2)\cdot r + (4\Delta \cdot W' - 2\Delta' \cdot W)) \\
&= (\alpha-1)\cdot[\Upsilon_1 \cdot r + \Upsilon_0] + [\mathrm{X}_1 \cdot r + \mathrm{X}_0]
\end{aligned} \tag{10.1}
$$

where

$$
\begin{aligned}
\Upsilon_1 &\equiv \Delta' \cdot (4W + \Delta' \cdot e_1) & \mathrm{X}_1 &\equiv 4\Delta \cdot W' - 2\Delta' \cdot W + (2\Delta \cdot \Delta'' - (\Delta')^2)\cdot e_1 \\
\Upsilon_0 &\equiv (4W^2 - (\Delta')^2 e_2)] & \mathrm{X}_0 &\equiv ((\Delta')^2 - 2\Delta \cdot \Delta'')\cdot e_2
\end{aligned} \tag{10.2}
$$

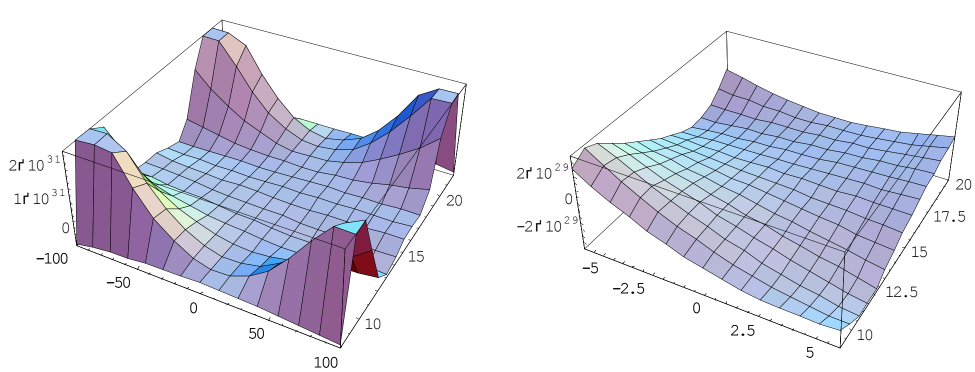
Since $$\underset{\sim}{M}(x,\alpha) \equiv \left(\frac{4\Delta^2 \cdot (r_1^\alpha)''}{\alpha \cdot r_1^{\alpha-2}}\right) \cdot \left(\frac{4\Delta^2 \cdot (r_2^\alpha)''}{\alpha \cdot r_2^{\alpha-2}}\right) \in \mathbb{Q}\{e_1, e_2\}[\alpha] \tag{10.3}$$

is zero if $(r_1^\alpha)''=0$ we may use it to determine when $(r_1^\alpha)''$ changes sign. The formula for $\underset{\sim}{M}(x,\alpha)$ in terms of $\Delta, \Delta', \Delta'', W, W', e_1, e_2$ obtained by multiplying out $\{(\alpha-1)\cdot[\Upsilon_1\cdot r_1+\Upsilon_0]+[\mathrm{X}_1\cdot r+\mathrm{X}_0]\}\cdot\{(\alpha-1)\cdot[\Upsilon_1\cdot r_2+\Upsilon_0]+[\mathrm{X}_1\cdot r+\mathrm{X}_0]\}$ does not immediately tell us upon inspection the location or even number of roots in $[\gamma_1,\gamma_5]$. So we left it to Mathematica to compute $\underset{\sim}{M}(x,\alpha)$ in terms of $e_1, e_2, e_1', e_2', e_1'', e_2''$. We first note that $\underset{\sim}{M}(x,\alpha)$ is quadratic in α.

Next we specialized $A\to 52369$, $B\to 73920$, $C\to 436041$, $D\to -1940670$, $E\to -5469210$, $F\to 25834841$, $e_1\to -\dfrac{Bx+E}{C}$, and $e_2\to \dfrac{Ax^2+Dx+F}{C}$. Then $\underset{\sim}{M}(x,\alpha)$ became octic in x. We factored out $\Delta(x)$ from $\underset{\sim}{M}(x,\alpha)$. Since $\Delta(x)>0, \forall x\in(\gamma_1,\gamma_5)$ it follows that $\underset{\sim}{M}(x,\alpha)\div\Delta(x)$, which we continue to denote as $\underset{\sim}{M}(x,\alpha)$ and is sextic in x, changes sign exactly where $\underset{\sim}{M}(x,\alpha)$ does.

We then created the standard Sturm sequence of $\underset{\sim}{M}$ and attempted to repeat the same type of procedure as in Section 8 for determining the subintervals of $\mathbb{R}$ and $[\gamma_1,\gamma_5]$ where α and x make $\underset{\sim}{M}(x,\alpha)\neq 0$. Unfortunately, the computation of the Sturm sequence worked once, but never again, on the Dell Win 98 computer. And the program was lost.

Many of the α-roots in Section 8 were found to be redundant anyway: they did not signal a change in total number of x-roots in $[\gamma_1,\gamma_5]$ for the terms of the resolvent. So instead, we made many 3D plots of $\underset{\sim}{M}(x,\alpha)$, like the ones in (10.4), to estimate these intervals.



(10.4)

We found that for $|\alpha|$ sufficiently large, about $|\alpha| \geq 6$, *very roughly*, $\underset{\sim}{M}(x,\alpha) < 0$ for $x \in [\gamma_1, 9]$, $\underset{\sim}{M}(x,\alpha) > 0$ for $x \in [9,11]$, $\underset{\sim}{M}(x,\alpha) < 0$ for $x \in [11,20]$, $\underset{\sim}{M}(x,\alpha) > 0$ for $x \in [20,\gamma_5]$. For $|\alpha| < 6$, $\underset{\sim}{M}(x,\alpha) < 0 \;\; \forall x \in [\gamma_1, \gamma_5]$. Since $(r_2^\alpha)'' > 0$, $\underset{\sim}{M}(x,\alpha)$ and $(r_1^\alpha)''$ have the same sign. □

Section 11. Kovacic's algorithm [3] for determining when the α-resolvent has algebraic solutions.

The differential equation $\frac{d^2 w}{dz^2} + a \cdot \frac{dw}{dz} + b \cdot w = 0$ can be transformed by the change of variables transformation $\upsilon \equiv w \cdot \exp\{\frac{1}{2}\int a \cdot dz\}$ to $\frac{d^2\upsilon}{dz^2} + J(z)\cdot \upsilon = 0$ where $J(z) \equiv b - \frac{1}{4}a^2 - \frac{1}{2}\frac{da}{dz}$. Suppose one were given the linear differential equation (2.3) with terms (3.3) with parameter α. How would one know that this differential equation were the α-resolvent of some polynomial?

Example 11.1. The α-resolvent (2.3) with terms (3.3) has algebraic solutions if and only if $\alpha \in \mathbb{Q}$.

Proof. Let $a = (F_{0,1} + F_{1,1}\cdot\alpha)/F_{0,2}$ and $b = (F_{1,0}\cdot\alpha + F_{2,0}\cdot\alpha^2)/F_{0,2}$. Since $\deg_z F_{0,2} = 5$, $\deg_z F_{0,1} = 4 = \deg_x F_{1,1}$, $\deg_z F_{1,0} = 3 = \deg_z F_{2,0}$ one can see by using the quotient rule of differentiation that $J = -\frac{Q_1}{Q_2}$ with $\deg_z Q_1 = 8$ and $\deg_z Q_2 = 10$. Therefore, the order of J at ∞ equals $10 - 8 = 2$. A computation by Mathematica of the partial fraction decomposition over the reals of $-J$ shows that it has the form (using Kovacic's notation, except for J, which is Ince's notation, since r is used elsewhere)

$$\frac{\alpha_1}{(z-\frac{2318581}{146523})^2} + \frac{\beta_1}{(z-\frac{2318581}{146523})} + \frac{\mu}{(z^2 - \frac{1465230}{48841}\cdot z + \frac{8615224}{48841})^2} + \frac{\kappa}{(z^2 - \frac{1465230}{48841}\cdot z + \frac{8615224}{48841})}$$

$$+ \frac{\chi}{(z^2 - \frac{1940670}{52369}\cdot z + \frac{25834841}{52369})^2} + \frac{\vartheta}{(z^2 - \frac{1940670}{52369}\cdot z + \frac{25834841}{52369})} \qquad (11.1)$$

where $\alpha_1 = \frac{3}{4}$ $\beta_1 = \frac{2^4\cdot 3^2\cdot 7\cdot 11\cdot 13^2\cdot 17^2}{43\cdot 488353139}$ $\mu = -\frac{3\cdot 7^2\cdot 48449}{2^2\cdot 13^2\cdot 17^2}$ $\chi = \frac{2^3\cdot 13^2\cdot 17^2\cdot 1052893}{52369^2}\cdot(1-\alpha^2)$

$$\kappa = \kappa_0 + \kappa_1 \cdot x = -\frac{1}{2^2 \cdot 5^2 \cdot 43^2 \cdot 488353139^2}\left(5^2 \cdot 271 \cdot 1588663 + 2^6 \cdot 3 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot z\right) \cdot \left(11^2 \cdot 9151999919 + 2^2 \cdot 7^2 \cdot 13^4 \cdot 17^4 \cdot \alpha^2\right)$$

$$\vartheta = \vartheta_0 + \vartheta_1 \cdot z = \frac{7^2 \cdot 13^4 \cdot 17^4}{5^2 \cdot 43^2 \cdot 52369 \cdot 488353139^2} \cdot (5^2 \cdot 23 \cdot 2467 \cdot 209218901 + 2^6 \cdot 3 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 52369 \cdot x) \cdot (\alpha^2 - 1) \tag{11.2}$$

Let $c_2 = d_2$ and $c_3 = d_3$ be the roots of $48841 \cdot z^2 - 1465230 \cdot z + 8615224 = 13^2 \cdot 17^2 \cdot z^2 - 2 \cdot 3 \cdot 5 \cdot 13^2 \cdot 17^2 \cdot z + 2^3 \cdot 1076903$, which are real. One can compute $(c_2 - c_3)^2 = \frac{2^2 \cdot 7^2 \cdot 48449}{13^2 \cdot 17^2}$ and $c_2 + c_3 = 2 \cdot 3 \cdot 5$. Let $c_4 = d_4$ and $c_5 = d_5$ be the roots of $52369 \cdot z^2 - 1940670 \cdot z + 25834841 = 52369 \cdot z^2 - 2^1 \cdot 3^2 \cdot 5^1 \cdot 21563^1 \cdot z + 71 \cdot 363871$, which are complex. One can compute $(c_4 - c_5)^2 = -\frac{2^5 \cdot 13^2 \cdot 17^2 \cdot 1052893}{52369^2}$ and $c_4 + c_5 = \frac{2 \cdot 3^2 \cdot 5 \cdot 21563}{52369}$. Then one can derive the partial fraction decomposition of J in Kovacic's notation

$$\begin{aligned} &\frac{\alpha_1}{(z-c_1)^2} + \frac{\beta_1}{z-d_1} + \frac{1}{d_2 - d_3} \cdot \left(\frac{\kappa_0 + \kappa_1 \cdot d_2}{z - d_2} - \frac{\kappa_0 + \kappa_1 \cdot d_3}{z - d_3}\right) + \frac{1}{d_4 - d_5} \cdot \left(\frac{\vartheta_0 + \vartheta_1 \cdot d_4}{z - d_4} - \frac{\vartheta_0 + \vartheta_1 \cdot d_5}{z - d_5}\right) \\ &+ \frac{\mu}{(c_2 - c_3)^2}\left(\frac{1}{(z-c_2)^2} + \frac{1}{(z-c_3)^2}\right) - \frac{2\mu}{(d_2 - d_3)^3}\left(\frac{1}{z-d_2} - \frac{1}{z-d_3}\right) + \\ &+ \frac{\chi}{(c_4 - c_5)^2}\left(\frac{1}{(z-c_4)^2} + \frac{1}{(z-c_5)^2}\right) - \frac{2\chi}{(d_4 - d_5)^3}\left(\frac{1}{z-d_4} - \frac{1}{z-d_5}\right) \end{aligned} \tag{11.3}$$

and therefore $\alpha_2 = \frac{\mu}{(c_2 - c_3)^2} = -\frac{3}{16} = \alpha_3$ and $\alpha_4 = \frac{\chi}{(c_4 - c_5)^2} = \frac{\alpha^2 - 1}{4} = \alpha_5$ and

$\beta_2 = \frac{(\kappa_0 + \kappa_1 \cdot d_2) \cdot (d_2 - d_3)^2 - 2\mu}{(d_2 - d_3)^3}$ and $\beta_3 = -\frac{(\kappa_0 + \kappa_1 \cdot d_3) \cdot (d_2 - d_3)^2 - 2\mu}{(d_2 - d_3)^3}$ and

$\beta_4 = \frac{(\vartheta_0 + \vartheta_1 \cdot d_4) \cdot (d_4 - d_5)^2 - 2\chi}{(d_4 - d_5)^3}$ and $\beta_5 = -\frac{(\vartheta_0 + \vartheta_1 \cdot d_5) \cdot (d_4 - d_5)^2 - 2\chi}{(d_4 - d_5)^3}$. With some algebra one sees that

$$\sum_{j=1}^{5} \beta_j = \beta_1 + \kappa_1 + \vartheta_1 = \frac{2^4 \cdot 3 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17^2}{5^2 \cdot 43^2 \cdot 488353139^2} \cdot \{3 \cdot 5^2 \cdot 43 \cdot 488353139 - (11^2 \cdot 9151999919 + 2^2 \cdot 7^2 \cdot 13^4 \cdot 17^4 \cdot \alpha^2) + 2^2 \cdot 7^2 \cdot 13^4 \cdot 17^4 \cdot (\alpha^2 - 1)\} = 0.$$

A necessary condition for algebraic solutions is that $\sqrt{1+4\alpha_i} \in \mathbb{Q}$ for each $i \in [5]$. One can compute $\sqrt{1+4\alpha_1} = 2 \in \mathbb{Q}$, $\sqrt{1+4\alpha_2} = \sqrt{1+4\alpha_3} = \sqrt{1-4\cdot\frac{3}{16}} = \sqrt{\frac{4}{16}} = \frac{1}{2} \in \mathbb{Q}$ and $\sqrt{1+4\alpha_4} = \sqrt{1+4\alpha_5} = \sqrt{1+4\frac{\alpha^2-1}{4}} = \sqrt{\alpha^2} = \alpha$. Thus a necessary condition for rational solutions is that $\alpha \in \mathbb{Q}$. With some algebra one can show

$$\sum_{j=1}^{5} \beta_j \cdot d_j = \beta_1 \cdot d_1 + \kappa_0 + \kappa_1 \cdot (d_2 + d_3) - \frac{2\mu}{(d_2 - d_3)^2} + \vartheta_0 + \vartheta_1 \cdot (d_4 + d_5) - \frac{2\chi}{(d_4 - d_5)^2} \text{ and}$$

$$\sum_{j=1}^{5} \alpha_j = \frac{2\mu}{(c_2 - c_3)^2} + \frac{2\chi}{(c_4 - c_5)^2} + \frac{3}{4} = \frac{4\alpha^2 - 1}{8} \text{ so}$$

$$\gamma \equiv \sum_{j=1}^{5} \alpha_j + \sum_{j=1}^{5} \beta_j \cdot d_j = \beta_1 \cdot d_1 + \frac{3}{4} + \kappa_0 + \kappa_1 \cdot (d_2 + d_3) + \vartheta_0 + \vartheta_1 \cdot (d_4 + d_5) \text{ because } c_j = d_j \text{ for } 2 \le j \le 5.$$

With help from Mathematica one computes

$$\begin{aligned}
\gamma &= \beta_1 \cdot \frac{2318581}{3\cdot 13^2 \cdot 17^2} + \frac{3}{4} + \kappa_0 + \kappa_1 \cdot 2\cdot 3\cdot 5 + \vartheta_0 + \vartheta_1 \cdot \frac{2\cdot 3^2 \cdot 5 \cdot 21563}{52369} \\
&= \frac{2^4\cdot 3^2\cdot 7\cdot 11\cdot 13^2\cdot 17^2}{43\cdot 488353139}\cdot\frac{2318581}{3\cdot 13^2\cdot 17^2} + \frac{3}{4} \\
&- \frac{271\cdot 1588663}{2^2\cdot 43^2\cdot 488353139^2}\cdot\left(11^2\cdot 9151999919 + 2^2\cdot 7^2\cdot 13^4\cdot 17^4\cdot\alpha^2\right) \\
&- \frac{2^4\cdot 3\cdot 7\cdot 11\cdot 13^2\cdot 17^2}{5^2\cdot 43^2\cdot 488353139^2}\cdot\left(11^2\cdot 9151999919 + 2^2\cdot 7^2\cdot 13^4\cdot 17^4\cdot\alpha^2\right)\cdot 2\cdot 3\cdot 5 \\
&+ \frac{7^2\cdot 13^4\cdot 17^4\cdot 23\cdot 2467\cdot 209218901}{43^2\cdot 52369\cdot 488353139^2}\cdot(\alpha^2-1) + \frac{2^6\cdot 3\cdot 7^3\cdot 11\cdot 13^6\cdot 17^6}{5^2\cdot 43^2\cdot 488353139^2}\cdot\frac{2\cdot 3^2\cdot 5\cdot 21563}{52369}\cdot(\alpha^2-1) = 0
\end{aligned} \tag{11.4}$$

So $\sqrt{1+4\gamma} = 1 \in \mathbb{Q}$. Therefore the α -resolvent (2.3) with terms (3.3) has algebraic solutions if and only if $\alpha \in \mathbb{Q}$. □

References

[1] Apostol, Tom M., *Calculus*, Vol. 2, 2nd ed., Blaisdell Publishing Co., Waltham, MA (1969).

[2] Jacobson, Nathan, *Basic Algebra I*, 2nd ed., W.H. Freeman & Co., New York (1985).

[3] Kovacic, Jerald J., *An algorithm for solving second order linear homogeneous differential equations,* Journal of Symbolic Computation **2** (1986), pp. 3-43.

[4] Nahay, John M., *Linear differential resolvents*, Doctoral dissertation, Rutgers University, New Jersey, 2000.

[5] ___________, *Powersum formula for polynomials whose distinct roots are differentially independent over constants*, Int. J. Math. Math. Sci. **32** (22 December 2002), no. 12, 721-738.

[6] ____________, *Linear relations among algebraic solutions of differential equations*, J. Differential Equations, **191**, (1 July 2003), no. 2, 323-347.

[7]_____________, *Powersum formula for differential resolvents*, Int. J. Math. Math. Sci. **2004** (1 February 2004), no. 7, 365-371.

[8] ___________, *Differential resolvents of minimal order and weight*, Int. J. Math. Math. Sci. **2004** (26 September 2004), no. 54, 2867-2894.

[9] ___________, *A partial factorization of the powersum formula*, Int. J. Math. Math. Sci. **2004** (16 October 2004), no. 58, 3075-3102.

[10] Wheeden, Richard L. & Antoni Zygmund, *Measure and Integral*, Marcel Dekker, New York (1977).

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