

The Kolchin Seminar in Differential Algebra

Finitely Generated Difference Field Extensions

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November 15, 2003

Let K be a difference field of zero characteristic with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, that is, a field K considered together with the mutually commuting injective endomorphisms $\{\alpha_1, \dots, \alpha_n\}$ of K . In this case K is also called a σ -field. If α_i are automorphisms, the σ -field K is called *inversive*. It is called *ordinary* if $n = 1$, and *partial* if $n > 1$.

If K_0 is a subfield of K and $\alpha(a) \in K_0$ for any $a \in K_0, \alpha \in \sigma$, we say that K_0 is a difference (or σ -) subfield of K and K is a difference (σ -) field extension of K_0 . We also say that we have a σ -field extension K/K_0 .

If $B \subseteq K$, then the intersection of all σ -subfields of K containing K_0 and B is called the difference (σ -) subfield of K generated by the set B over K_0 ; it is denoted by $K_0\langle B \rangle$. As a field, $K_0\langle B \rangle = K_0(\{\tau(b) | b \in B, \tau \in T\})$.

The set B is called the set of σ -generators of K/K_0 . If $|B| < \infty$, $B = \{b_1, \dots, b_k\}$, we say that K is a finitely generated difference (σ -) field extension of K_0 and write $K = K_0\langle b_1, \dots, b_k \rangle$.

The concept of limit degree of a finitely generated ordinary difference field extension was introduced by R. Cohn (1956) as follows.

Let K be an ordinary difference ring with a basic set $\sigma = \{\alpha\}$ and $L = K\langle S \rangle$ a σ -field extension of K generated by a finite set S .

Let S_k denote the set $\{\alpha^i(s) | s \in S, 0 \leq i \leq k\}$ and $d_k = K(S_k) : K(S_{k-1})$ for $k = 1, 2, \dots$ ($S_0 = S$). Then $d_k = \alpha(K)(\alpha(S_k)) : \alpha(K)(\alpha(S_{k-1})) \geq K(S \cup \alpha(S_k)) : K(S \cup \alpha(S_{k-1})) = d_{k+1}$ for $k = 1, 2, \dots$

Let $d(S) = \min\{d_k | k = 1, 2, \dots\}$ if some d_k is finite, or $d(S) = \infty$ if all d_k are infinite.

Lemma 1 With the above notation, $d(S)$ does not depend on the systems of difference generators S of L/K .

This lemma shows that $d(S)$ is a characteristic of the extension L/K . It is called the *limit degree* of L/K and denoted by $ld(L/K)$.

If L/K is not finitely generated, its limit degree $ld(L/K)$ is defined to be the maximum of the limit degrees of all finitely generated difference subextensions of L/K , if this maximum exists, or ∞ if it does not.

PROOF. Let $L = K\langle S \rangle = K\langle S' \rangle$ where S and S' are two different finite subsets of L .

Let $d = d(S)$ and $e = d(S')$. Obviously, there exists $p \in \mathbf{N}$ such that $K(S) \subseteq K(S'_p)$ and $K(S') \subseteq K(S_p)$, hence $K(S_k) \subseteq K(S'_{k+p})$ and $K(S'_k) \subseteq K(S_{k+p})$ for $k = 1, 2, \dots$. Let $q > p$. If r is sufficiently large, then $K(S_{r+k+1}) : K(S_{r+k}) = d$ and $K(S'_{r+k+1}) : K(S'_{r+k}) = e$ for $k = 0, 1, \dots$. Therefore, $K(S_{r+p+q}) : K(S_r) = d^{p+q}$ and $K(S'_{r+q}) : K(S'_{r+p}) = e^{q-p}$

Since $K(S'_{r+q}) \subseteq K(S_{r+q+p})$ and $K(S_r) \subseteq K(S'_{r+p})$, we obtain that $K(S'_{r+q}) : K(S'_{r+p}) \leq K(S_{r+q+p}) : K(S_r)$, that is, $e^{q-p} \leq d^{q+p}$ or $(e/d)^q \leq (ed)^p$ for all $q > p$. Letting $q \rightarrow \infty$ we obtain that $e/d \leq 1$, that is, $e \leq d$. Similarly we get $d \leq e$ hence $e = d$.

The following result, that gives the main property of limit degree, is due to R. Cohn.

Theorem 1 *Let K be an ordinary difference field with a basic set σ , M a σ -overfield of K and L/K a σ -field subextension of M/K . Then $ld(M/K) = [ld(M/L)][ld(L/K)]$.*

(The proof is given in the Appendix for a similar result for partial difference fields.)

In the case of fields of characteristic $p > 0$, one can define an analog of limit degree using separable factor of degree in place of degree of field extensions. The corresponding invariant of the difference field extension is called the *reduced limit degree* of L/K and denoted by $rl d L/K$; it also has the multiplicative property established for $ld L/K$ in Theorem 1.

Theorem 2 *Let K be an ordinary difference (σ -) field and let L be a σ -field extension of K .*

(i) *If the difference field extension L/K is finitely generated, then $ld(L/K) = 1$ if and only if $L = K(S)$ for some finite set $S \subseteq L$.*

(ii) *The following statements are equivalent:*

- (a) *L/K is finitely generated, L is algebraic over K , and $ld(L/K) = 1$.*
- (b) *$L : K$ is finite.*

The concept of limit degree of ordinary field extensions plays the key role in the results on compatibility of difference field extensions. Two difference field extensions L/K and M/K of the same difference field K are called *incompatible* if they cannot be embedded into a common difference field extension of K .

Example 1 (R. Cohn) Let us consider \mathbf{Q} as an ordinary difference field whose basic set σ consists of the identity automorphism α . If one adjoins to \mathbf{Q} an element i such that $i^2 = -1$, then the result field $\mathbf{Q}(i)$ has two automorphisms that extend α : one of them is the identical mapping (we denote it by the same letter α) and the other (denoted by β) sends an element $a+bi \in \mathbf{Q}(i)$ ($a, b \in \mathbf{Q}$) to $a-bi$ (complex conjugation). Then $\mathbf{Q}(i)$ can be treated as a difference field with the basic set $\{\alpha\}$, as well as a difference field with the basic set $\{\beta\}$. Denoting these two difference fields by G and H , respectively, we can naturally consider them as σ -field extensions of \mathbf{Q} . Let us show that G/\mathbf{Q} and H/\mathbf{Q} are incompatible σ -field extensions. Indeed, suppose that there exists a σ -field extension E of \mathbf{Q} and σ -isomorphisms ϕ and ψ , respectively, of G/\mathbf{Q} and H/\mathbf{Q} into E/\mathbf{Q} . Let $j = \phi(i)$, $k = \psi(i)$, and let γ denote the translation of E that extends α and β . Then $j^2 = k^2 = -1$ whence either $j = k$ or $j = -k$. Since $\gamma(j) = j$ and $\gamma(k) = -k$, in both cases we obtain that $j = -j$, that is, $j = 0$. This contradiction implies that G/\mathbf{Q} and H/\mathbf{Q} are incompatible.

Let L be a σ -field extension of an ordinary difference (σ -) field K . The *core* L_K of L over K is defined to be the set of elements $a \in L$ algebraic and separable over K and such that $ld(K\langle a \rangle/K) = 1$.

It follows from Theorem 1 that L_K is a σ -field and $ld L_K/K = 1$. Example 1 and Theorem 2 show that L_K need not to be K , and $L = L_K$ if and only if $L : K < \infty$.

The following is an example of a theorem on compatibility where the concept of core plays the key role.

Theorem 3 *Let K be an ordinary difference field with a basic set σ and let L and M be two σ -field extensions of K . Then the following statements are equivalent.*

- (i) L/K and M/K are incompatible.
- (ii) There exist finitely generated σ -field extensions L' and M' of K such that $L' \subseteq L$, $M' \subseteq M$, and L'/K and M'/K are incompatible.
- (iii) L_K/K and M_K/K are incompatible.
- (iv) L_K/K and M/K are incompatible.

The following three results give an alternative descriptions of the core.

Let K be an ordinary inversive difference field of zero characteristic with a basic set $\sigma = \{\alpha\}$ and let L be a difference field extension of K such that L/K is algebraic (in the usual sense). Furthermore, let L_K denote the core of L over K .

Theorem 4 (R. Cohn). *With the above conventions, let $a \in L$. Then $a \in L_K$ if and only if $a \in K\langle\alpha(a)\rangle$.*

Theorem 5. Let K and L be as in Theorem 4 and let $L = K\langle S \rangle$ where S is a finite subset of L . Then

$$L_K = \bigcap_{n=0}^{\infty} K\langle\alpha^n(S)\rangle.$$

Theorem 6. Let K and $L = K\langle S \rangle$ ($\text{Card } S < \infty$) be as in Theorem 5, and let $ld L/K = K(S) : K$. Then $\bigcap_{n=0}^{\infty} K\langle\alpha^n(S)\rangle = K$, that is, $L_K = K$.

Another important application of limit degree is the R. Cohn's proof of the fundamental theorem on finitely generated field extension.

Theorem 7. *Let K be an ordinary difference field with a basic set $\sigma = \{\alpha\}$, M a finitely generated σ -field extension of K , and L and intermediate difference field of M/K . Then the σ -field extension L/K is finitely generated.*

PROOF. *Step 1.* Let $\{\eta_1, \dots, \eta_r\}$ be a difference transcendence basis of M/L and let $L' = L\langle\eta_1, \dots, \eta_r\rangle$. Then the fact that L'/K is finitely generated implies that L/K is finitely generated.

Indeed, let $L' = K\langle\lambda_1, \dots, \lambda_m\rangle$. Then each λ_i is a quotient of two polynomials in $\alpha^j\eta_k$ with coefficients in L . Let S be the set of all these coefficients. We are going to show that $L = K\langle S \rangle$. If $\gamma \in L$, then $\gamma \in L'$, so there exist two polynomials P and Q in $\alpha^j\eta_k$ with coefficients in $K\langle S \rangle$ such that $Q\gamma = P$. Since the set of all $\alpha^j\eta_k$ is algebraically independent over L , the coefficients of the corresponding products of $\alpha^j\eta_k$ in $Q\gamma$ and P must be equal. Since $\gamma \in L$ and all the coefficients of P and Q belong to $K\langle S \rangle$, we obtain that $\gamma \in K\langle S \rangle$, so $L = K\langle S \rangle$.

Step 2. Let B be a transcendence basis of L'/K . Since the difference transcendence degree $\sigma\text{-trdeg}(L'/K) \leq \sigma\text{-trdeg}(M/K) < \infty$, the set B is finite. Therefore, in order to prove that L/K is finitely generated, it is sufficient to prove that $L'/K\langle B \rangle$ is finitely generated. Since

$\sigma\text{-trdeg}(M/K\langle B \rangle) = \sigma\text{-trdeg}(M/L) + \sigma\text{-trdeg}(L'/K\langle B \rangle) = 0 + 0 = 0$, we can assume that $\sigma\text{-trdeg}(M/K) = 0$.

Step 3. Since $\sigma\text{-trdeg}(M/K) = 0$, $ld(M/K) < \infty$ hence $ld(L/K) < \infty$. Therefore, there exists a finite set $\Phi \subseteq L$ such that $ld(L/K) = ld(K\langle \Phi \rangle/K)$, hence $ld(L/K\langle \Phi \rangle) = 1$. Clearly, it suffices to prove that $L/K\langle \Phi \rangle$ is finitely generated, so we can now assume that $\sigma\text{-trdeg}(M/K) = 0$ and $ld(L/K) = 1$ (therefore, $ld(M/K) = ld(M/L)$).

Step 4. Let $M = K\langle V \rangle$ where $Card V < \infty$. Let $V_t = V \cup \alpha(V) \cup \dots \cup \alpha^t(V)$ ($t \in \mathbf{N}$). Since $ld(M/K) = ld(M/L)$, there exists $p \in \mathbf{N}$ such that $K(V_{p+h+1}) : K(V_{p+h}) = L(V_{p+h+1}) : L(V_{p+h}) = ld(M/L) < \infty$ for every $h \in \mathbf{N}$. Let us show that $L \subseteq K(V_p)$.

Suppose that there exists $\eta \in L \setminus K(V_p)$. since $\bigcup_{h=0}^{\infty} K(V_{p+h}) = M \supseteq L$, there exists $q \in \mathbf{N}$ such that $\eta \in K(V_{p+q+1})$, $\eta \notin K(V_{p+q})$. Then $ld(M/K) = K(V_{p+q+1}) : K(V_{p+q}) > K(V_{p+q+1}) : K(\{\eta\} \cup V_{p+q}) \geq L(V_{p+q+1}) : L(V_{p+q}) = ld(M/L) = ld(M/K)$, a contradiction.

Thus, $L \subseteq K(V_p)$ hence $L = K(V')$ for some finite set V' hence $L = K\langle V' \rangle$. This completes the proof.

In 1984 P. Evanovich (*Finitely generated extensions of partial difference fields*. Trans. AMS, 281, no. 2, 795 - 811) introduced an invariant $ld_n(M/K)$ of a partial difference field extension M/K that can be viewed as a generalization of the concept of limit degree.

$ld_n(M/K)$ is inductively defined as an element of the set $\mathbf{N} \cup \{\infty\}$ that satisfies the following conditions (ld1) - (ld5). Let K be a difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, M a σ -overfield of K and L/K is a σ -field subextension of M/K . Then

(ld1) *If $M = L\langle S \rangle$ for a finite set $S \subseteq M$, then there exists a finitely generated σ -overfield K' of K contained in L such that $ld_n(M/L) = ld_n(K'\langle S \rangle/K')$.*

(ld2) *If $S \subseteq M$, then $ld_n(L\langle S \rangle/K\langle S \rangle) \leq ld_n(L/K)$ and $ld_n(L\langle S \rangle/L) \leq ld_n(K\langle S \rangle/K)$. Equality will hold in both if S is σ -algebraically independent over L .*

(ld3) *If there is a σ -isomorphism ϕ of L onto a σ -field L' and K' is a σ -subfield of L' such that $\phi(K) = K'$, then $ld_n(L/K) = ld_n(L'/K')$.*

(ld4) If the σ -field extension L/K is finitely generated, then $\sigma\text{-trdeg}_k L = 0$ if and only if $ld_n(L/K) < \infty$.

(ld5) $ld_n(M/K) = ld_n(M/L) \cdot ld_n(L/K)$.

If $n = 1$, ld_1 is defined to be the limit degree ld for ordinary difference fields. Suppose that ld_{n-1} is defined for difference (σ -) field extensions with $\text{Card } \sigma = n - 1$. Let L/K be a finitely generated difference field extension with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, $L = K\langle S \rangle$ for a finite set $S \subseteq L$. For any $m \in \mathbf{N}$, let $L_m = K\langle\langle \cup_{i=1}^m \alpha_n^i(S) \rangle\rangle_{\sigma'}$ where $\sigma' = \{\alpha_1, \dots, \alpha_{n-1}\}$. Then L_m/L_{m-1} is a finitely generated σ' -field extension. Applying (ld3) and (ld2) we obtain that $ld_{n-1}(L_m/L_{m-1}) \geq ld_{n-1}(L_{m+1}/L_m)$, hence there exists $\lim_{m \rightarrow \infty} ld_{n-1}(L_m/L_{m-1}) = a$ where $a \in \mathbf{N}$ or $a = \infty$.

As in the case of ordinary difference fields, one can show that a is independent on the choice of σ -generators of L/K . Now we define $ld_n(L/K) = a$.

If L/K is not finitely generated, $ld_n(L/K)$ is defined to be the maximum of $ld_n(K'/K)$ where K'/K is a finitely generated σ -field subextension of L/K , if the maximum exists and ∞ if it does not. Using the induction on ld_n , P. Evanovich proved the following analog of Theorem 7 for partial difference fields.

Theorem 7* *Let K be a difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, M a finitely generated σ -overfield of K and L/K is a σ -field subextension of M/K . Then the σ -field extension L/K is finitely generated.*

The Evanovich's approach to the limit degree of partial difference field extensions can be simplified as follows.

Let K be a difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$ and let T denote the free commutative multiplicative semigroup generated by the elements $\alpha_1, \dots, \alpha_n$. Let us consider the well-ordering \preceq of T such that $\tau = \alpha_1^{k_1} \dots \alpha_n^{k_n} \preceq \tau' = \alpha_1^{l_1} \dots \alpha_n^{l_n}$ if and only if $(k_n, \dots, k_1) \leq_{lex} (l_n, \dots, l_1)$. Furthermore, for any $r_1, \dots, r_n \in \mathbf{N}$, we set $T(r_1, \dots, r_n) = \{\tau \in T \mid \tau \preceq \alpha_1^{r_1} \dots \alpha_n^{r_n}\}$. The last notation will be also used with the symbol ∞ instead for some r_i (with the condition $k < \infty$ for any $k \in \mathbf{N}$).

Let L be a difference (σ -) field extension of a difference field K generated by a finite family S , that is, $L = K\langle S \rangle$. As a field, $L = K(\{\tau(S) \mid \tau \in T\})$ where $\tau(S)$ denotes the set $\{\tau(a) \mid a \in S\}$.

If $(r_1, \dots, r_n) \in \mathbf{N}^n$ and $r_1 \geq 1$, then we set $d(S; r_1, \dots, r_n) = K(T(r_1, \dots, r_n)(S)) : K(T(r_1 - 1, \dots, r_n)(S))$. ($d(S; r_1, \dots, r_n)$ is either a non-negative integer or ∞ .)

Lemma 2 *With the above notation,*

$d(S; r_1, \dots, r_n) \geq d(S; r_1 + p_1, \dots, r_n + p_n)$ for every $(p_1, \dots, p_n) \in \mathbf{N}^n$.

Lemma 3 *Let K be a difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, and let S and S' be two finite systems of generators of a difference (σ -) field extension L/K , that is, $L = K\langle S \rangle = K\langle S' \rangle$. Then $d(S) = d(S')$.*

Lemma 3 shows that $d(S)$ does not depend on the system of generators S of L/K . It is called the *limit degree* of L/K and denoted by $ld(L/K)$. If L/K is not finitely generated, then its limit degree $ld(L/K)$ is defined as the maximum of limit degrees of finitely generated difference subextensions N/K ($K \subseteq N \subseteq L$) if this maximum exists, or ∞ if it does not.

It is easy to check that our definition of the limit degree is equivalent to the P. Evanovich's definition, but it is explicit and it allows one to prove statements about partial difference field extensions using the ideas from the ordinary case. In particular, modifying the proofs for the ordinary difference fields, one can prove the multiplicative property of limit degree and the fundamental theorem on finitely generated partial difference field extensions (the proof is essentially shorter than the proof in the paper by Evanovich).

In conclusion we are going to give a short proof of the fundamental theorem based on the properties of characteristic sets of difference polynomials.

Let K be a difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, T the free commutative semigroup generated by σ , and $R = K\{y_1, \dots, y_s\}$ the algebra of difference (σ -) polynomials. (R can be viewed as a polynomial ring in the set of indeterminates $TY = \{\tau y_i \mid \tau \in T, 1 \leq i \leq s\}$ over K .) Elements of the set TY are called *terms*.

By a *ranking* of the set TY we mean a well-ordering \leq of this set satisfying

(i) $u \leq \tau u$ for any $u \in TY, \tau \in T$.

(ii) If $u, v \in TY$ and $u \leq v$, then $\tau u \leq \tau v$ for any $\tau \in T$.

Example: $u = \alpha_1^{k_1} \dots \alpha_n^{k_n} y_i \leq v = \alpha_1^{l_1} \dots \alpha_n^{l_n} y_j \in TY$ if and only if $(\sum_{\nu=1}^n k_\nu, i, k_1, \dots, k_n)$ is less than or equal to $(\sum_{\nu=1}^n l_\nu, j, l_1, \dots, l_n)$ with respect to the lexicographic order on \mathbf{N}^{n+2} . In what follows, we fix a ranking of TY .

Let $A \in K\{y_1, \dots, y_s\}$. The greatest (with respect to the given ranking) element of TY that appears in A is called the *leader* of A ; it is denoted by u_A . If A is written as a polynomial in u_A , $A = \sum_{i=0}^d I_i u_A^i$ ($d = \deg_{u_A} A$ and the σ -polynomials I_0, \dots, I_d do not contain u_A), then I_d is called the *initial* of the σ -polynomial A ; it is denoted by I_A .

Let $A, B \in K\{y_1, \dots, y_s\}$. We say that A has lower rank than B and write $A < B$, if either $A \in K, B \notin K$ or $u_A < u_B$ or $u_A = u_B = u$, $\deg_u A < \deg_u B$. If neither $A < B$ nor $B < A$, we say that A and B have the same rank and write $rk A = rk B$. Furthermore, we say that A is *reduced* with respect to B if A does not contain any power of a transform τu_B ($\tau \in T$) whose exponent is greater than or equal to $\deg_{u_B} B$. If S is any subset of $K\{y_1, \dots, y_s\} \setminus K$, then a σ -polynomial $A \in K\{y_1, \dots, y_s\}$ is said to be reduced with respect to S if A is reduced with respect to every element of S .

A set $\Sigma \subseteq K\{y_1, \dots, y_s\}$ is called an *autoreduced set* if either $\Sigma = \emptyset$ or $\Sigma \cap K = \emptyset$ and every element of Σ is reduced with respect to all others. It is easy to see that distinct elements of an autoreduced set have distinct leaders. As in the case of differential polynomials, one can show that every autoreduced set is finite.

Theorem 8 *Let $\mathcal{A} = \{A_1, \dots, A_p\}$ be an autoreduced set in $K\{y_1, \dots, y_s\}$. Let $I(\mathcal{A}) = \{B \in K\{y_1, \dots, y_s\} \mid \text{either } B = 1 \text{ or } B \text{ is a product of finitely many } \sigma\text{-polynomials of the form } \tau(I_{A_i}) \text{ } (\tau \in T, i = 1, \dots, p)\}$. Then for any $C \in K\{y_1, \dots, y_s\}$, there exist σ -polynomials $J \in I(\mathcal{A})$ and $C_0 \in K\{y_1, \dots, y_s\}$ such that C_0 is reduced with respect to \mathcal{A} and $JC \equiv C_0 \pmod{[\mathcal{A}]}$ (that is, $JC - C_0 \in [\mathcal{A}]$).*

The reduction process, that is, a transition from C to C_0 can be performed as follows.

If C is reduced with respect to \mathcal{A} , we can take $C_0 = C$ and $J = 1$. If C is not reduced with respect to \mathcal{A} , then C contains a power $(\tau u_{A_i})^k$ of some term τu_{A_i} ($\tau \in T_\sigma, 1 \leq i \leq p$) whose exponent is greater than or equal to $\deg_{u_{A_i}} A_i$. Such a term τu_{A_i} of the highest possible rank is called the \mathcal{A} -*leader* of C and denoted by $v_{\mathcal{A},C}$. Obviously, C can be written as $C = Dv_{\mathcal{A},C}^d + E$

where D does not contain $v_{\mathcal{A},C}$ and $\deg_{v_{\mathcal{A},C}}E < d$. Let $v_{\mathcal{A},C} = \tau u_{A_j}$ ($\tau \in T_\sigma, 1 \leq j \leq p$). Then $v_{\mathcal{A},C}$ is the leader of τA_j , $I_{\tau A_j} = \tau I_{A_j}$, and $\deg_{v_{\mathcal{A},C}}(\tau A_j) = d_j$ where $d_j = \deg_{u_{A_j}}A_j$. Consider the σ -polynomial $C' = (\tau I_{A_j})C - v_{\mathcal{A},C}^{d-d_j}\tau A_j D$. Clearly, $v_{\mathcal{A},C'} \leq v_{\mathcal{A},C}$ and in the case of equality, $\deg_{v_{\mathcal{A},C'}} < d_j$. Furthermore, $C' \equiv C \pmod{[\mathcal{A}]}$. Applying the same procedure to C' and continuing this process, we obtain a σ -polynomial \bar{C} such that $\bar{C} \equiv C \pmod{[\mathcal{A}]}$ and $v_{\mathcal{A},\bar{C}} < v_{\mathcal{A},C}$. Repeating the foregoing procedure we obtain a σ -polynomial C_0 that satisfies the conditions of the theorem.

Let $\mathcal{A} = \{A_1, \dots, A_p\}$ and $\mathcal{B} = \{B_1, \dots, B_q\}$ be two autoreduced sets whose elements are written in the order of increasing rank. We say that \mathcal{A} has lower rank than \mathcal{B} if one of the following conditions holds:

- (i) *there exists $k \in \mathbf{N}$, $1 \leq k \leq \min\{p, q\}$, such that $rk A_i = rk B_i$ for $i = 1, \dots, k-1$ and $A_k < B_k$;*
- (ii) *$p > q$ and $rk A_i = rk B_i$ for $i = 1, \dots, q$.*

As in the case of differential polynomials, one can prove that every nonempty set of autoreduced sets contains an autoreduced set of lowest rank. If J is a difference ideal of $K\{y_1, \dots, y_s\}$, an autoreduced subset of J of lowest rank is called a *characteristic set* of J .

It is easy to prove that if Σ is a characteristic set of J , then J does not contain nonzero difference polynomials reduced with respect to Σ . In particular, if $A \in \Sigma$, then $I_A \notin J$.

Theorem 9 *Let K be a difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$ and let $M = K\langle \eta_1, \dots, \eta_s \rangle$ be a difference (σ -) field extension of K generated by a finite family $\{\eta_1, \dots, \eta_s\}$. Furthermore, let L be an intermediate σ -field of the extension M/K . Then L is a finitely generated σ -field extension of K .*

PROOF. Let P be the defining difference ideal of the s -tuple $\eta = (\eta_1, \dots, \eta_s)$ in the algebra of σ -polynomials $L\{y_1, \dots, y_s\}$ over L , and let $\Sigma = \{A_1, \dots, A_p\}$ be a characteristic set of the σ -ideal P . Let $I(\Sigma)$ denote a subset of $L\{y_1, \dots, y_s\}$ consisting of 1 and all finite products of σ -polynomials of the form $\tau(I_A)$ where $A \in \Sigma$, $\tau \in T$. Furthermore, let Φ be the set of all coefficients of all σ -polynomials of Σ and let $K' = K\langle \Phi \rangle$. Obviously, K' is a finitely generated σ -field extension of K containing in L . We are going to prove the theorem by showing that $K' = L$.

Suppose that $\lambda \in L \setminus K'$. Since $\lambda \in M = K\langle\eta_1, \dots, \eta_s\rangle$, there exist $A, B \in K\{y_1, \dots, y_s\}$ such that $\lambda = A(\eta)/B(\eta)$. Then $A(\eta) - \lambda B(\eta) = 0$ whence $C = A - \lambda B \in P$. It follows that C reduces to zero modulo $[\Sigma]$, that is, there exists $J \in I(\Sigma)$ such that JC is a linear combination of σ -polynomials $\tau(A_i)$ ($\tau \in T$, $1 \leq i \leq p$) with coefficients $D_{\tau i}$ of the form $D_{\tau i} = D'_{\tau i} + \lambda D''_{\tau i}$ where $D'_{\tau i}, D''_{\tau i} \in K'\langle\eta_1, \dots, \eta_s\rangle$.

In other words, $JC = H_1 + \lambda H_2 = 0$ where H_1 and H_2 are linear combinations of elements $\tau(A_i)$ ($\tau \in T$, $1 \leq i \leq p$) with coefficients in $K'\langle\eta_1, \dots, \eta_s\rangle$. Moreover, it is easy to see that H_1 is the result of the reduction of A with respect to Σ , so that $H_1 \neq 0$ (otherwise $A \in P$ and $\lambda = A(\eta)/B(\eta) = 0$). Now the equality $H_1 = \lambda H_2$ and the fact that $H_1, H_2 \in K'\langle\eta_1, \dots, \eta_s\rangle$ imply that $\lambda \in K'$. This completes the proof of the theorem.

APPENDIX

1. An approach to the limit degree of partial difference equations

Let K be a difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$ and let T denote the free commutative multiplicative semigroup generated by the elements $\alpha_1, \dots, \alpha_n$.

In what follows we shall consider two orders on T : the product order \leq_P such that $\tau = \alpha_1^{k_1} \dots \alpha_n^{k_n} \leq_P \tau' = \alpha_1^{l_1} \dots \alpha_n^{l_n}$ if and only if $k_i \leq l_i$ for $i = 1, \dots, n$, and the order \preccurlyeq such that $\tau \preccurlyeq \tau'$ if and only if $(k_n, \dots, k_1) <_{lex} (l_n, \dots, l_1)$. Obviously, T is well-ordered with respect to \preccurlyeq .

For any $r_1, \dots, r_n \in \mathbf{N}$, we set $P(r_1, \dots, r_n) = \{\tau = \alpha_1^{k_1} \dots \alpha_n^{k_n} \in T \mid 0 \leq k_i \leq r_i \text{ (} i = 1, \dots, n)\}$ and $T(r_1, \dots, r_n) = \{\tau \in T \mid \tau \preccurlyeq \alpha_1^{r_1} \dots \alpha_n^{r_n}\}$. The last notation will be also used with the symbol ∞ instead for some r_i (with the condition $k < \infty$ for any $k \in \mathbf{N}$).

Let L be a difference (σ -) field extension of a difference field K generated by a finite family S , that is, $L = K\langle S \rangle$. As a field, $L = K(\{\tau(S) \mid \tau \in T\})$ where $\tau(S)$ denotes the set $\{\tau(a) \mid a \in S\}$. (More generally, if $T' \subseteq T$, then $T'S$ will denote the set $\{\tau(a) \mid \tau \in T', a \in S\}$.) If $\sigma' \subset \sigma$, then K can be naturally treated as a difference field with a basic set σ' . A σ' -field extension of this σ' -field generated by a set S will be denoted by $K\langle S \rangle_{\sigma'}$.

If $(r_1, \dots, r_n) \in \mathbf{N}^n$ and $r_1 \geq 1$, then the degree $K(T(r_1, \dots, r_n)(S)) : K(T(r_1-1, \dots, r_n)(S))$ will be denoted by $d(S; r_1, \dots, r_n)$. (Obviously, $d(S; r_1, \dots, r_n)$ is either a non-negative integer or ∞ .)

In what follows we shall need the following statement from the classical field theory.

Lemma 1 *Let M/K be a field extension and L an intermediate field of M/K .*

(i) *If $S \subseteq M$, then $L(S) : K(S) \leq L : K$ and $L(S) : L \leq K(S) : K$. The equalities hold if S is an algebraically independent set over L .*

(ii) *If $M = L(\Sigma)$ for some finite set Σ , then there exists a finitely generated field extension K' of K such that $K' \subseteq L$ and $M : L = K'(\Sigma) : K'$.*

Lemma 2 *With the above notation, $d(S; r_1, \dots, r_n) \geq d(S; r_1 + p_1, \dots, r_n + p_n)$ for every $(p_1, \dots, p_n) \in \mathbf{N}^n$.*

PROOF. Clearly, it is sufficient to prove that $d(S; r_1, \dots, r_n) \geq d(S; r_1 + 1, \dots, r_n)$ and $d(S; r_1, \dots, r_n) \geq d(S; r_1, \dots, r_{i-1}, r_i + 1, r_{i+1}, \dots, r_n)$ for every $i = 2, \dots, n$.

To prove the first inequality, notice that $d(S; r_1, \dots, r_n) = \alpha_1(K)(\alpha_1 T(r_1, \dots, r_n)S) : \alpha_1(K)(\alpha_1 T(r_1 - 1, \dots, r_n)S) \geq K(\alpha_1 T(r_1, \dots, r_n)S) \cup T(0, r_2, \dots, r_n)S) : K(\alpha_1 T(r_1 - 1, \dots, r_n)S) \cup T(0, r_2, \dots, r_n)S = K(T(r_1 + 1, r_2, \dots, r_n)(S)) : K(T(r_1, \dots, r_n)(S))$ (we use the inequalities from Lemma 1 (i)), so that $d(S; r_1, \dots, r_n) \geq d(S; r_1 + 1, \dots, r_n)$.

Similarly, if $2 \leq i \leq n$, then $d(S; r_1, \dots, r_n) = \alpha_i(K)(\alpha_i T(r_1, \dots, r_n)S) : \alpha_i(K)(\alpha_i T(r_1 - 1, r_2, \dots, r_n)S) \geq K(\alpha_i T(r_1, \dots, r_n)S) \cup T(\infty, \dots, \infty, 0, r_{i+1}, \dots, r_n)S) : K(\alpha_i T(r_1 - 1, r_2, \dots, r_n)S) \cup T(\infty, \dots, \infty, 0, r_{i+1}, \dots, r_n)S = d(S; r_1, \dots, r_{i-1}, r_i + 1, r_{i+1}, \dots, r_n)$. This completes the proof.

The last lemma implies that if $d(S; r_1, \dots, r_n)$ is finite for some $(r_1, \dots, r_n) \in \mathbf{N}^n$, then $\min\{d(S; r_1, \dots, r_n) \mid (r_1, \dots, r_n) \in \mathbf{N}^n\}$ is finite. In this case we denote this minimum value of $d(S; r_1, \dots, r_n)$ by $d(S)$. If $d(S; r_1, \dots, r_n) = \infty$ for all $(r_1, \dots, r_n) \in \mathbf{N}^n$, we set $d(S) = \infty$. The following statement shows that $d(S)$ does not depend on the system of generators S of L/K . Therefore, $d(S)$ can be considered as a characteristic of this finitely generated difference extension; it is called the *limit degree* of L/K and denoted by $ld(L/K)$. If a difference field extension L/K is not finitely generated, then its limit degree $ld(L/K)$ is defined as the maximum of limit degrees of finitely generated difference subextensions N/K ($K \subseteq N \subseteq L$) if this maximum exists, or ∞ if it does not. As it follows from the multiplicative property of limit

degree proved below, if a difference field extension L/K is finitely generated, then $ld(L/K)$ is also the maximum of the limit degrees of the finitely generated subextensions of L/K (including L/K itself.)

Lemma 3 *Let K be a difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, and let S and S' be two finite systems of generators of a difference (σ -) field extension L/K , that is, $L = K\langle S \rangle = K\langle S' \rangle$. Then $d(S) = d(S')$.*

PROOF. Let $d = d(S)$ and $e = d(S')$ ($i = 1, \dots, n$). As before, the degrees $K(T(r_1, \dots, r_n)(S)) : K(T(r_1 - 1, \dots, r_n)S)$ and $K(T(r_1, \dots, r_n)S') : K(T(r_1 - 1, r_2, \dots, r_n)S')$ will be denoted by $d(S; r_1, \dots, r_n)$ and $d(S'; r_1, \dots, r_n)$, respectively.

It is easy to see that there exists a positive integer h such that $K(S) \subseteq K(T(h, \dots, h)S')$ and $K(S') \subseteq K(T(h, \dots, h)S)$. Since $T(k_1, \dots, k_n)T(l_1, \dots, l_n) = T(k_1 + l_1, \dots, k_n + l_n)$ for any $(k_1, \dots, k_n), (l_1, \dots, l_n) \in \mathbf{N}^n$, the last two inclusions imply that

$$K(T(r_1, \dots, r_n)S) \subseteq K(T(r_1 + h, \dots, r_n + h)S') \quad (1)$$

and

$$K(T(r_1, \dots, r_n)S') \subseteq K(T(r_1 + h, \dots, r_n + h)S) \quad (2)$$

for any $(r_1, r_2, \dots, r_n) \in \mathbf{N}^n$.

Assume, first, that $d < \infty$ and $e < \infty$. Then there exists $m \in \mathbf{N}$ such that $d = K(T(m, \dots, m)S) : K(T(m - 1, m, \dots, m)S)$ and $e = K(T(m, \dots, m)S') : K(T(m - 1, m, \dots, m)S')$ whence

$$d = K(T(m + p_1, \dots, m + p_n)S) : K(T(m + p_1 - 1, m + p_2, \dots, m + p_n)S) \quad (3)$$

and

$$e = K(T(m + p_1, \dots, m + p_n)S) : K(T(m + p_1 - 1, m + p_2, \dots, m + p_n)S) \quad (4)$$

for every $(p_1, \dots, p_n) \in \mathbf{N}^n$.

Therefore, for every integer $k > h$ we have

$$K(T(m + k + h, \dots, m + k + h)S) : K(T(m, \dots, m)S) = d^{n(k+h)} \quad (5)$$

and

$$K(T(m+k, \dots, m+k)S') : K(T(m+h, \dots, m+h)S') = e^{n(k-h)}. \quad (6)$$

Furthermore, the inclusions (1) and (2) imply that

$$K(T(m+k, \dots, m+k)S') \subseteq K(T(m+k+h, \dots, m+k+h)S) \quad (7)$$

and

$$K(T(m, \dots, m)S) \subseteq K(T(m+h, \dots, m+h)S'). \quad (8)$$

Combining (7) and (8) with the equalities (5) and (6) we obtain that

$$d^{n(k+h)} \geq e^{n(k-h)} \quad (9)$$

for all $k > h$. Allowing $k \rightarrow \infty$ in the last inequality we arrive at the inequality $d \geq e$. Similarly we can obtain that $e \geq d$ whence $e = d$. Furthermore, inclusions (7) and (8) (and inclusions obtained from (7) and (8) by interchanging S and S') imply that if one of the values $d(S)$, $d(S')$ is finite, then the other value is also finite and $d(S) = d(S')$. This completes the proof of the lemma.

Let K be a difference field with a basic set σ and L a σ -field extension of K . We say that an element $v \in L$ is *transformally dependent* or *σ -algebraically dependent on a set $A \subseteq L$ over K* if v is σ -algebraic over the field $K\langle A \rangle$ (that is, the set $\{\tau(v) \mid \tau \in T\}$ is algebraically dependent over $K\langle A \rangle$). Obviously, an element $v \in L$ is σ -algebraically dependent on a set $A \subseteq L$ if and only if there exists a finite family $\{\eta_1, \dots, \eta_s\} \subseteq A$ such that v is σ -algebraic over $K\langle \eta_1, \dots, \eta_s \rangle$.

In what follows we present some statements about σ -algebraic dependence that are used in the study of limit degree. The proofs of these statements can be found in [1].

Proposition 4 *Let L/K be a difference (σ -) field extension and $A \subseteq L$. Then:*

(i) *The set A is σ -algebraically dependent over K if and only if there exists $v \in A$ such that v is σ - (respectively, σ^* -) algebraically dependent on $A \setminus \{v\}$ over K .*

(ii) *The set A contains a maximal subset σ -algebraically independent over K . In other words, there exists a set $B \subseteq A$ such that B is σ -algebraically independent over K and any subset of A containing B is σ -algebraically dependent over K .*

A set B , whose existence is established by Proposition 4 (ii), is called a *basis for transformal transcendence* or a *difference (or σ -) transcendence basis of A over K* . If $A = L$, the set B is called a *basis for transformal transcendence* or a *difference (or σ -) transcendence basis of L over K* .

Proposition 5 *Let L/K be a difference (σ -) field extension, let B and B' be two subsets of L and $v, u_1, \dots, u_m \in L$. Then:*

(i) *If v is σ -algebraically dependent on B over K and every element of B is σ -algebraically dependent on B' over K , then v is σ -algebraically dependent on B' over K .*

(ii) *If v is σ -algebraically dependent on $\{u_1, \dots, u_m\}$, but not on $\{u_1, \dots, u_{m-1}\}$ over K , then u_m is σ -algebraically dependent on the set $\{u_1, \dots, u_{m-1}, v\}$ over K .*

(iii) *Suppose that $B' \subseteq B$, $u_1, \dots, u_m \in B$ are σ -algebraically independent over K , and each u_i ($1 \leq i \leq m$) is σ -algebraically dependent on B' over K . Then there exist elements $v_1, \dots, v_m \in B'$ such that each v_i is σ -algebraically dependent over K on the set B'' obtained from B' by replacing v_j by u_j ($j = 1, \dots, m$).*

Proposition 6 *Let L/K be a difference (σ -) field extension and $A \subseteq L$.*

(i) *Suppose that B is a subset of A which is σ -algebraically independent over K . Then B is a σ -transcendence basis of A over K if and only if every element of A is σ -algebraically dependent on B over K .*

(ii) *All σ -transcendence bases of A over K either contain the same finite number of elements or are infinite.*

With the notation of the last proposition, the **difference (or σ -) transcendence degree** of A over K is defined as the number of elements of any σ -transcendence basis of A over K , if this number is finite, or infinity in the contrary case.

The σ -transcendence degree of A over K is denoted by $\sigma\text{-trdeg}_K A$. In particular, if $A = L$, then $\sigma\text{-trdeg}_K L$ denotes the σ -transcendence degree of the σ -field extension L/K .

Proposition 7 *Let K be a difference field with a basic set σ and L a σ -field extension of K .*

(i) *Any family of σ -generators of L over K contains a σ -transcendence basis of this difference field extension.*

(ii) Let $\eta_1, \dots, \eta_m \in L$. Then $\sigma\text{-trdeg}_K K\langle \eta_1, \dots, \eta_m \rangle \leq m$.

(iii) Let $\{\eta_1, \dots, \eta_m\}$ and $\{\zeta_1, \dots, \zeta_s\}$ be two finite subsets of L such that $K\langle \eta_1, \dots, \eta_m \rangle = K\langle \zeta_1, \dots, \zeta_s \rangle$. If the set $\{\zeta_1, \dots, \zeta_s\}$ is σ -algebraically independent over K , then $s \leq m$.

Proposition 8 Let $H \subseteq K \subseteq L$ be difference field extensions with the same basic set σ . Then $\sigma\text{-trdeg}_H L = \sigma\text{-trdeg}_K L + \sigma\text{-trdeg}_H K$.

The following lemma gives some relationships between the limit degree and difference transcendence degree.

Lemma 9 Let K be a difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$ and L a σ -field extension of K . Furthermore, let $\sigma_{n-1} = \{\alpha_1, \dots, \alpha_n\}$.

(i) If $\sigma\text{-trdeg}_K L > 0$, then $ld(L/K) = \infty$.

(ii) If the σ -field extension L/K is finitely generated and $\sigma\text{-trdeg}_K L = 0$, then $ld(L/K) < \infty$.

PROOF. (i) Suppose that $\sigma\text{-trdeg}_K L > 0$, but $ld(L/K) = d < \infty$. Then L contains an element η which is σ -transcendental over K . Obviously, $ld(K\langle \eta \rangle/K) < \infty$, so there exists $(r_1, \dots, r_n) \in \mathbf{N}^n$ such that $K(T(r_1, \dots, r_n)\eta) : K(T(r_1 - 1, r_2, \dots, r_n)\eta) < \infty$. In this case $\alpha_1^{r_1} \dots \alpha_n^{r_n} \eta$ is algebraic over the field $K(T(r_1 - 1, \dots, r_n)\eta)$ that contradicts the fact that η is σ -transcendental over K .

(ii) Let $L = K\langle S \rangle$ for some finite set $S \subseteq L$ and let $\sigma\text{-trdeg}_K L = 0$. Assume, first that $\text{Card } S = 1$, $S = \{\eta\}$. The order \preccurlyeq on T naturally determines a well-ordering of the terms τy ($\tau \in T$) in the ring of difference polynomials $K\{y\}$ in one difference indeterminate y . (Using the same symbol \preccurlyeq for this well-ordering, we have $\tau y \preccurlyeq \tau' y$ if and only if $\tau \preccurlyeq \tau'$.) Since η is σ -algebraic over K , there exists a difference polynomial $P \in K\{y\}$ such that $P(\eta) = 0$ ($P(\eta)$ is obtained by replacing every τy in P by $\tau \eta$). Let $u = \tau y$ be the greatest term of P with respect to \preccurlyeq , and let $\tau = \alpha_1^{r_1} \dots \alpha_n^{r_n}$ where $r_1, \dots, r_n \in \mathbf{N}$, $r_1 \geq 1$ (if $r_1 = 1$, we can replace P by $\alpha_1 P$). Then the element $\tau(\eta)$ is algebraic over $K(T(r_1 - 1, \dots, r_n)\eta)$ hence $K(T(r_1, \dots, r_n)\eta) : K(T(r_1 - 1, \dots, r_n)\eta) < \infty$, hence $ld(L/K) < \infty$.

Now let $\text{Card } S > 1$, $S = \{\eta_1, \dots, \eta_m\}$. As we have seen, for every $i = 1, \dots, m$, there exist $r_{i1}, \dots, r_{in} \in \mathbf{N}$ such that $K(T(r_{i1}, \dots, r_{in})\eta_i) : K(T(r_{i1} - 1, \dots, r_{in})\eta_i) < \infty$. By Lemma 2, there exists a sufficiently large $r \in \mathbf{N}$ such that $K(T(r, \dots, r)\eta_i) : K(T(r - 1, r, \dots, r)\eta_i) < \infty$

for $i = 1, \dots, m$. Then $K(\cup_{i=1}^m T(r, \dots, r)\eta_i) : K(\cup_{i=1}^m T(r-1, r, \dots, r)\eta_i) < \infty$ hence $ld(L/K) < \infty$.

Theorem 10 *Let K be a difference field with a basic set σ , M a σ -field extension of K and L an intermediate σ -field of this extension. Then $ld(M/K) = [ld(M/L)][ld(L/K)]$.*

PROOF. If $\sigma\text{-trdeg}_K M > 0$, then $ld(M/K) = \infty$ (see Lemma 9) and at least one of the numbers $\sigma\text{-trdeg}_L M$, $\sigma\text{-trdeg}_K L$ is positive, so the corresponding limit degree is ∞ and the statement of the theorem is true. Thus, we can assume that $\sigma\text{-trdeg}_K M = 0$, so that every element of M is σ -algebraic over K .

Suppose, first, that L and M are finitely generated σ -field extensions of K . Let $L = K\langle X \rangle$, $M = K\langle Y \rangle$ (X and Y are some finite sets), and let $d = ld(L/K)$, $e = ld(M/L)$, and $f = ld(M/K)$. (Because of our assumption, d , e , and f are finite). Obviously, $M = K\langle X \cup Y \rangle$ and there exist $p_1, \dots, p_n \in \mathbf{N}$ such that

$$d = K(T(r_1, \dots, r_n)X) : K(T(r_1 - 1, r_2, \dots, r_n)X),$$

$$e = L(T(r_1, \dots, r_n)Y) : L(T(r_1 - 1, r_2, \dots, r_n)Y), \text{ and}$$

$$f = K(T(r_1, \dots, r_n)(X \cup Y)) : K(T(r_1 - 1, r_2, \dots, r_n)(X \cup Y))$$

for all r_1, \dots, r_n with $(p_1, \dots, p_n) \leq_P (r_1, \dots, r_n)$. (\leq_P denotes the product order on \mathbf{N}^n). It follows that for any $(h_1, \dots, h_n) \in \mathbf{N}^n$,

$$K(T(r_1 + h_1, \dots, r_n + h_n)X) : K(T(r_1, \dots, r_n)X) = d^{h_1 + \dots + h_n},$$

$$L(T(r_1 + h_1, \dots, r_n + h_n)Y) : L(T(r_1, \dots, r_n)Y) = e^{h_1 + \dots + h_n}, \text{ and}$$

$$K(T(r_1 + h_1, \dots, r_n + h_n)(X \cup Y)) : K(T(r_1, \dots, r_n)(X \cup Y)) = f^{h_1 + \dots + h_n}.$$

Let us show that there exists a finitely generated field (not necessarily a σ -field) subextension N/K of L/K such that

$$N(T(p_1 + 1, p_2, \dots, p_n)Y) : N(T(p_1, \dots, p_n)Y) = e. \quad (10)$$

Indeed, applying Lemma 1(ii) to the sequence of field extensions

$K((T(p_1, \dots, p_n)Y) \subseteq L((T(p_1, \dots, p_n)Y) \subseteq L(T(p_1 + 1, \dots, p_n)Y)$, we obtain that there exists a finite set $W \subseteq L(T(p_1, \dots, p_n)Y)$ such that $K(W \cup T(p_1 + 1, \dots, p_n)Y) : K(W \cup T(p_1, \dots, p_n)Y) = L(T(p_1 + 1, \dots, p_n)Y) : L(T(p_1, \dots, p_n)Y) = e$.

Since $W \subseteq L((T(p_1, \dots, p_n)Y) = K(\cup_{\tau \in T} \tau(X) \cup T(p_1, \dots, p_n)Y)$, there exists a finite set $\Lambda \subseteq L$ such that $W \subseteq K(\Lambda \cup (T(p_1, \dots, p_n)Y)$. By Lemma 1(i), $e = L(T(p_1 + 1, p_2, \dots, p_n)Y) :$

$$\begin{aligned}
& L(T(p_1, \dots, p_n)Y) \leq K(\Lambda \cup T(p_1 + 1, p_2, \dots, p_n)Y) : K(\Lambda \cup T(p_1, \dots, p_n)Y) \leq K(W \cup T(p_1 + \\
& 1, p_2, \dots, p_n)Y) : K(W \cup T(p_1, \dots, p_n)Y) = e. \text{ Setting } N = K(\Lambda) \text{ we satisfy condition (10).} \\
& \text{Furthermore, since } \Lambda \subseteq K\langle X \rangle, \text{ there exist positive integers } q_1, \dots, q_n \text{ such that } (p_1, \dots, p_n) \leq_P \\
& (q_1, \dots, q_n) \text{ and } N \subseteq K(T(q_1, \dots, q_n)X). \text{ Applying Lemma 1(i) we obtain that} \\
& K(T(q_1 + 1, q_2, \dots, q_n)X \cup T(p_1 + 1, p_2, \dots, p_n)Y) : K(T(q_1 + 1, q_2, \dots, q_n)X \cup T(p_1, \dots, p_n)Y) \leq \\
& N(T(p_1 + 1, p_2, \dots, p_n)Y) : N(T(p_1, \dots, p_n)Y) = e
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
& K(T(q_1 + 1, q_2, \dots, q_n)X \cup T(p_1, \dots, p_n)Y) : K(T(q_1, \dots, q_n)X) \cup T(p_1, \dots, p_n)Y \\
& \leq K(T(q_1 + 1, q_2, \dots, q_n)X) : K(T(q_1, \dots, q_n)X) \leq d.
\end{aligned} \tag{12}$$

Combining (11) and (12) we arrive at the inequality

$$K(T(q_1 + 1, q_2, \dots, q_n)X) \cup T(p_1 + 1, p_2, \dots, p_n)Y : K(T(q_1, \dots, q_n)X) \cup T(p_1, \dots, p_n)Y \leq de. \tag{13}$$

Let $U = T(q_1, \dots, q_n)X \cup T(p_1, \dots, p_n)Y$. Since $X \cup Y \subseteq U$, $M = K\langle U \rangle$. Therefore,

$$\begin{aligned}
& f \leq K(U \cup \alpha_1(U)) : K(U) = K(T(q_1 + 1, \dots, q_n)X \cup T(p_1 + 1, p_2, \dots, p_n)Y) \\
& : K(T(q_1, \dots, q_n)X \cup T(p_1, \dots, p_n)Y) \leq de.
\end{aligned} \tag{14}$$

In order to prove the opposite inequalities $f \geq de$, let us consider a transcendence basis B of the set $T(p_1, \dots, p_n)Y$ over $K(T(p_1, \dots, p_n)X)$ such that $B = \cup_{i=1}^n B_i$ where B_1 is a transcendence basis of $T(p_1, \dots, p_n)Y$ over $K(T(p_1, \dots, p_n)X \cup T(0, p_2, \dots, p_n)Y)$ and $B_i \subseteq T(0, \dots, 0, p_i, \dots, p_n)Y$ is a transcendence basis of $K(T(p_1, \dots, p_n)X \cup T(0, \dots, 0, p_i, \dots, p_n)Y)$ over $K(T(p_1, \dots, p_n)X \cup T(0, \dots, 0, p_{i-1}, \dots, p_n)Y)$ ($i = 2, \dots, n$). It is easy to see that the degree $m = K(T(p_1, \dots, p_n)X \cup T(p_1, \dots, p_n)Y) : K(T(p_1, \dots, p_n)X \cup B)$ is finite. Furthermore, if $h \in \mathbf{N}$, then every element of the set $T(p_1 + h, p_2, \dots, p_n)X$ is algebraic over $K(T(p_1, \dots, p_n)X)$ (it follows from the fact that $d < \infty$). Therefore, for any $h \in \mathbf{N}$, the set B is algebraically independent over $K(T(p_1 + h, p_2, \dots, p_n)X)$ and

$$\begin{aligned}
& K(T(p_1 + h, p_2, \dots, p_n)X \cup B) : K(T(p_1, \dots, p_n)X \cup B) = \\
& K(T(p_1 + h, p_2, \dots, p_n)X) : K(T(p_1, \dots, p_n)X) = d^h.
\end{aligned} \tag{15}$$

Now, using the inclusions $K(T(p_1, \dots, p_n)X \cup B) \subseteq K(T(p_1, \dots, p_n)X \cup T(p_1, \dots, p_n)Y)$ and $\cup T(p_1, \dots, p_n)Y \subseteq K(T(p_1 + h, p_2, \dots, p_n)X \cup T(p_1, \dots, p_n)Y)$ and

$$K(T(p_1, \dots, p_n)X \cup B) \subseteq K(T(p_1+h, p_2, \dots, p_n)X \cup B) \subseteq K(T(p_1+h, p_2, \dots, p_n)X \cup T(p_1, \dots, p_n)Y)$$

we obtain that

$$\begin{aligned} & [K(T(p_1+h, p_2, \dots, p_n)X \cup T(p_1, \dots, p_n)Y) : K(T(p_1, \dots, p_n)X \cup T(p_1, \dots, p_n)Y)] \\ & \cdot [K(T(p_1+h, p_2, \dots, p_n)X \cup B) : K(T(p_1, \dots, p_n)X \cup B)] \geq \\ & K(T(p_1+h, p_2, \dots, p_n)X \cup B) : K(T(p_1, \dots, p_n)X \cup B). \end{aligned}$$

The last inequality, together with (15), implies that

$$\begin{aligned} & K(T(p_1+h, p_2, \dots, p_n)X \cup T(p_1, \dots, p_n)Y) : K(T(p_1, \dots, p_n)X \cup T(p_1, \dots, p_n)Y) \\ & \geq \frac{d^h}{m} \end{aligned} \quad (16)$$

Furthermore, Lemma 1(ii) yields the inequality

$$\begin{aligned} & K(T(p_1+h, p_2, \dots, p_n)X \cup T(p_1+h, p_2, \dots, p_n)Y) : K(T(p_1+h, p_2, \dots, p_n)Y \cup T(p_1, \dots, p_n)X) \geq \\ & L(T(p_1+h, p_2, \dots, p_n)Y) : L(T(p_1, \dots, p_n)Y) = e^h. \end{aligned} \quad (17)$$

Combining inequalities (16) and (17) we obtain that

$$\begin{aligned} & f^h = K(T(p_1+h, p_2, \dots, p_n)X \cup T(p_1+h, p_2, \dots, p_n)Y) : \\ & K(T(p_1, \dots, p_n)Y \cup T(p_1, \dots, p_n)Y) \geq \frac{d^h e^h}{m} \end{aligned} \quad (18)$$

for all $h \in \mathbf{N}$. If we let $h \rightarrow \infty$, we will see that the last inequality can hold only if $de \leq f$. Thus, $f = de$ for $i = 1, \dots, n$ that completes the proof of the theorem for the case when L/K and M/K are finitely generated difference field extensions.

Now suppose that $K \subseteq L \subseteq M$ is any chain of difference fields and $\sigma\text{-trdeg}_K M = 0$. (As we have seen, the general case can be reduced to the case with the last condition.) Let d, e and f be as before. Then the first part of the proof shows that if X and Y are finite subsets of L and M , respectively, then

$$f \geq ld(K\langle X \cup Y \rangle / K) = [ld(K\langle X \cup Y \rangle / K\langle X \rangle)][ld(K\langle X \rangle / K)]. \quad (19)$$

It is clear that $K\langle X \rangle(T(r_1+1, r_2, \dots, r_n)Y) : K\langle X \rangle(T(r_1, \dots, r_n)Y) \geq L(T(r_1+1, r_2, \dots, r_n)Y) : L(T(r_1, \dots, r_n)Y)$ for any r_1, \dots, r_n whence $ld(L\langle Y \rangle / L) \leq ld(K\langle X \cup Y \rangle / K\langle X \rangle)$. This inequality, together with (19), implies that

$$d \geq [ld(L\langle Y \rangle / L)][ld(K\langle X \rangle / K)]. \quad (20)$$

If $d < \infty$ and $e < \infty$, one can choose X and Y such that the factors in the right-hand side of (20) are d and e . If either d or e is ∞ , one can choose X or Y to make the corresponding factor arbitrarily large. In either case we obtain that $f \geq de$.

In order to prove the opposite inequality (and complete the proof of the theorem), it is sufficient to show that $de \geq f$ in the case $d < \infty$, $e < \infty$ (otherwise, $de = \infty \geq f$). In other words, we should prove that if U is a finite subset of M , then

$$ld(K\langle U\rangle/K) \leq de. \quad (21)$$

If M/L is finitely generated, then the first part of the proof shows that $ld(L\langle U\rangle/L) \leq ld(M/L) = e$. If M/L is not finitely generated, then we also have $ld(L\langle U\rangle/L) \leq e$ by the definition of $ld(M/L)$. It follows that there exist $r_1, \dots, r_n \in \mathbf{N}$ such that $L(T(r_1 + 1, r_2, \dots, r_n)U) : L(T(r_1, \dots, r_n)U) \leq e$. Now, as in the first part of the proof, we obtain that there exists a finite set $P \subseteq L$ such that $K(P \cup T(r_1 + 1, r_2, \dots, r_n)U) : K(P \cup T(r_1, \dots, r_n)U) \leq e$. Therefore,

$$ld(K\langle P \cup U\rangle/K\langle P\rangle) \leq e. \quad (22)$$

From the first part of the proof, if L/K is finitely generated, or by the definition of $ld(L/K)$ otherwise, we have

$$ld(K\langle P\rangle/K) \leq ld(L/K) = d. \quad (23)$$

Using the already proven statement for finitely generated extensions and (22), (23), we can evaluate $ld(K\langle P \cup U\rangle/K)$ in two ways as follows:

$$ld(K\langle P \cup U\rangle/K) = [ld(K\langle P \cup U\rangle/K\langle P\rangle)][ld(K\langle P\rangle/K)] \leq de \quad (24)$$

and

$$ld(K\langle P \cup U\rangle/K) = [ld(K\langle P \cup U\rangle/K\langle U\rangle)][ld(K\langle U\rangle/K)] \quad (25)$$

Combining (24) and (25) we obtain that $ld(K\langle U\rangle/K) \leq de$. This completes the proof of the theorem.

2. On the core of a purely algebraic ordinary difference field extension

In what follows we keep the notation used in [1]. In particular, if G is an overfield of a field F , then $G : F$ denotes the dimension of G as a vector space over F ; if $S \subset G$, then $F(S)$ denotes the smallest field containing F and S . If G is a difference overfield of F and $S \subset G$, then $F\langle S \rangle$ denotes the smallest difference field containing F and S . As usual, \mathbf{Z} , \mathbf{N} and \mathbf{Q} denote the sets of all integers, all non-negative integers and all rational numbers, respectively.

Let K be an ordinary inversive difference field of zero characteristic with a basic set $\sigma = \{\alpha\}$. Let L be a purely algebraic difference field extension of K (that is, every element of L is algebraic over K) and let L_K denote the core of L over K . In other words, L_K is the set of all elements $a \in L$ such that $ld K\langle a \rangle / K = 1$. It is known that L_K is an inversive difference overfield of K and $ld L_K / K = 1$ (see [1, p.215]).

Theorem 1 (R. Cohn). *With the above conventions, let $a \in L$. Then $a \in L_K$ if and only if $a \in K\langle \alpha(a) \rangle$.*

PROOF. Let $a \in L_K$. Then $K\langle a \rangle : K < \infty$ hence there exists $r \in \mathbf{N}$ such that $K\langle a \rangle = K(a, \alpha(a), \dots, \alpha^r(a))$. Since α is an automorphism of the field K , $K(\alpha(a), \alpha^2(a), \dots, \alpha^{r+1}(a)) : K = K(a, \alpha(a), \dots, \alpha^r(a)) : K = K\langle a \rangle : K$. It follows that $K(\alpha(a), \alpha^2(a), \dots, \alpha^{r+1}(a)) = K\langle a \rangle$ whence $a \in K\langle \alpha(a) \rangle$.

Conversely, suppose that $a \in K\langle \alpha(a) \rangle$. Then there exists $n \in \mathbf{N}$ such that $a \in K(\alpha(a), \dots, \alpha^{n+1}(a))$. It follows that

$$K(a, \alpha(a), \dots, \alpha^{n+1}(a)) = K(\alpha(a), \dots, \alpha^{n+1}(a)). \quad (1)$$

Since α is an automorphism of the field K , $K(a, \alpha(a), \dots, \alpha^n(a)) : K = K(\alpha(a), \dots, \alpha^{n+1}(a)) : K$. Furthermore, equality (1) shows that $K(\alpha(a), \dots, \alpha^{n+1}(a)) \supseteq K(a, \alpha(a), \dots, \alpha^n(a))$. Therefore,

$$K(\alpha(a), \dots, \alpha^{n+1}(a)) = K(a, \alpha(a), \dots, \alpha^n(a)), \quad (2)$$

whence $K(a, \alpha(a), \dots, \alpha^n(a)) = K(a, \alpha(a), \dots, \alpha^n(a), \alpha^{n+1}(a))$.

It follows that $ld K\langle a \rangle / K = 1$ and $a \in L_K$. □

Theorem 2. Let K and L be as in Theorem 1 and let $L = K\langle S \rangle$ where S is a finite subset of L . Then

$$L_K = \bigcap_{n=0}^{\infty} K\langle \alpha^n(S) \rangle. \quad (3)$$

PROOF, Let $a \in L_K$. Repeatedly applying Theorem 1 we obtain that $a \in K\langle \alpha^n(a) \rangle$ for $n = 1, 2, \dots$, so that $a \in \bigcap_{n=0}^{\infty} K\langle \alpha^n(a) \rangle$. Since $a \in K\langle S \rangle$, $K\langle \alpha^n(a) \rangle \subseteq K\langle \alpha^n(S) \rangle$ for all $n \in \mathbf{N}$ whence $\bigcap_{n=0}^{\infty} K\langle \alpha^n(a) \rangle \subseteq \bigcap_{n=0}^{\infty} K\langle \alpha^n(S) \rangle$ and $a \in \bigcap_{n=0}^{\infty} K\langle \alpha^n(S) \rangle$. Thus, $L_K \subseteq \bigcap_{n=0}^{\infty} K\langle \alpha^n(S) \rangle$.

Now let us prove the opposite inclusion. Without loss of generality we can assume that the set S consists of a single element b (that is, $L = K\langle b \rangle$) and $K(b, \alpha(b)) : K = d$ where d denotes the limit degree of L over K . Indeed, let $ld L/K = K(S, \alpha(S), \dots, \alpha^{m+1}(S)) : K(S, \alpha(S), \dots, \alpha^m(S))$ for some $m \in \mathbf{N}, m \geq 1$. By the theorem on a primitive element, there exists an element $b \in K(S, \alpha(S), \dots, \alpha^m(S))$ such that $K(b) = K(S, \alpha(S), \dots, \alpha^m(S))$. It is easy to see that $K\langle b \rangle = K\langle S \rangle = L$, $K(b, \alpha(b)) = K(S, \alpha(S), \dots, \alpha^{m+1}(S))$ (hence $K(b, \alpha(b)) : K(b) = d$), and $\bigcap_{n=0}^{\infty} K\langle \alpha^n(b) \rangle = \bigcap_{n=0}^{\infty} K\langle \alpha^n(S) \rangle$. (Clearly, $K\langle \alpha^n(b) \rangle = K\langle \alpha^n(S) \rangle$ for $n = 1, 2, \dots$)

Let s denote the degree of b over K , that is, the degree of the minimal polynomial of b in the polynomial ring $K[X]$. Then every element $\alpha^i(b)$ ($i \in \mathbf{N}$) is algebraic of degree s over K . Furthermore, for any $t \in \mathbf{N}, t \geq 1$, $K(\alpha^i(b), \alpha^{i+1}(b), \dots, \alpha^{i+t}(b)) : K(\alpha^i(b), \alpha^{i+1}(b), \dots, \alpha^{i+t-1}(b)) = K(b, \alpha(b), \dots, \alpha^t(b)) : K(b, \alpha(b), \dots, \alpha^{t-1}(b)) = d$ hence $K(\alpha^i(b), \alpha^{i+1}(b), \dots, \alpha^{i+t}(b)) : K = sd^t$ and the set $\{\alpha^i(b)^{k_0} \alpha^{i+1}(b)^{k_1} \dots \alpha^{i+t}(b)^{k_t} \mid 0 \leq k_0 \leq s, 0 \leq k_\nu \leq d-1 \text{ for } \nu = 1, \dots, t\}$ is a basis of the vector K -space $K(\alpha^i(b), \alpha^{i+1}(b), \dots, \alpha^{i+t}(b))$.

Now, suppose that $a \in \bigcap_{n=0}^{\infty} K\langle \alpha^n(b) \rangle$. Since $a \in K\langle b \rangle$, there exists $r \in \mathbf{N}$ such that $a \in K(b, \alpha(b), \dots, \alpha^r(b))$, so that a is a finite linear combination with coefficients from K of elements of the form $b^{k_0} \alpha(b)^{k_1} \dots \alpha^r(b)^{k_r}$ where $0 \leq k_0 \leq s$ and $0 \leq k_\nu \leq d-1$ for $\nu = 1, \dots, r$. At the same time, if $p \in \mathbf{N}, p \geq r+1$, then $a \in K\langle \alpha^p(b) \rangle$, hence $a \in K(\alpha^p(b), \alpha^{p+1}(b), \dots, \alpha^{p+q}(b))$ for some non-negative integer q . We assume that q is the smallest such a number.

The last inclusion shows that a can be represented as a linear combination with coefficients from K of elements of the form $\alpha^p(b)^{k_0} \alpha^{p+1}(b)^{k_1} \dots \alpha^{p+q}(b)^{k_q}$ where $0 \leq k_0 \leq s$ and

$0 \leq k_\nu \leq d - 1$ for $\nu = 1, \dots, q$. Comparing the two representations of a we obtain that if $q \geq 1$, then $\alpha^{p+q}(b)$ is a root of a polynomial of degree at most $d - 1$ with coefficients from the field $K' = K(b, \alpha(b), \dots, \alpha^{p+q-1}(b))$. This contradicts the fact that $K'(\alpha^{p+q}(b)) : K' = d$, that is, the minimal polynomial of $\alpha^{p+q}(b)$ over K' has degree d . Therefore, $q = 0$ and $a \in K(\alpha^p(b))$. We arrive at the inclusion $a \in \bigcap_{j=p+1}^{\infty} K(\alpha^j(b))$ which implies that $\alpha(a) \in \bigcap_{j=p+2}^{\infty} K(\alpha^j(b))$, $\alpha^2(a) \in \bigcap_{j=p+3}^{\infty} K(\alpha^j(b))$, \dots . Since $K(\alpha^{p+s+1}(b)) : K = s$ and $a, \alpha(a), \dots, \alpha^s(a) \in \bigcap_{j=p+s+1}^{\infty} K(\alpha^j(b)) \subseteq K(\alpha^{p+s+1}(b))$, elements $a, \alpha(a), \dots, \alpha^s(a)$ are linearly dependent over K . It follows that there exists the smallest positive integer k such that the elements $a, \alpha(a), \dots, \alpha^k(a)$ are linearly dependent over K . Then $\alpha^k(a)$ is a linear combination of $a, \alpha(a), \dots, \alpha^{k-1}(a)$ over K hence $K(a, \alpha(a), \dots, \alpha^k(a)) = K(a, \alpha(a), \dots, \alpha^{k-1}(a))$. It follows that $ld K\langle a \rangle / K = 1$ and $a \in L_K$. This completes the proof of the theorem. \square

Example. Let $K = \mathbf{Q}(x, y_0, y_1, y_{-1}, y_2, y_{-2}, \dots)$ be the field of rational fractions in a denumerable set of indeterminates $x, y_0, y_1, y_{-1}, y_2, y_{-2}, \dots$ over \mathbf{Q} . Let us consider K as an inversive ordinary difference field with a basic set $\sigma = \{\alpha\}$ where α acts on \mathbf{Q} as the identity mapping, $\alpha(x) = x$, and $\alpha(y_r) = y_{r+1}$ for every $r \in \mathbf{Z}$. Let u and t_i ($i \in \mathbf{N}$) denote \sqrt{x} and $\sqrt{y_i}$, respectively, and let $L = K(u, t_0, t_1, t_{-1}, \dots)$. (Thus, the field L is obtained by adjoining to K a root of the polynomial $X^2 - x \in K[X]$ and the denumerable set of roots of the polynomials $X^2 - y_i$, $i \in \mathbf{Z}$.)

The field L can be naturally considered as a difference field extension of K where $\alpha(u) = u$ and $\alpha(t_i) = t_{i+1}$ for $i = 0, 1, 2, \dots$. It is easy to see that the set $S = \{u, t_0\}$ generates this σ -field extension, $L = K\langle S \rangle = K\langle u + t_0 \rangle$ and $K\langle \alpha^n(S) \rangle = K\langle u + t_n \rangle$ for $n = 0, 1, 2, \dots$. In this case $L_K = \bigcap_{n=0}^{\infty} K\langle \alpha^n(S) \rangle = \bigcap_{n=0}^{\infty} K\langle u + t_n \rangle = K(u)$. (With the notation of the proof of the last theorem, $b = u + t_0$, $s = 4$, and $d = 2$.)

The proof of the following statement uses the ideas of the proof of Theorem 2 and [1, Example 5, p. 219].

Theorem 3. Let K and $L = K\langle S \rangle$ ($Card S < \infty$) be as in Theorem 2, and let $ld L/K = K(S) : K$. Then $\bigcap_{n=0}^{\infty} K\langle \alpha^n(S) \rangle = K$, that is, $L_K = K$.

PROOF. As in the proof of Theorem 2, without loss of generality we can assume that the set S consists of a single element b such that $K(b) : K = K(b, \alpha(b)) : K(b) = d$ where $d = ld L/K$. Furthermore, the arguments of the same proof show that the set $B = \{\alpha^{i_1}(b)^{k_1} \dots \alpha^{i_m}(b)^{k_m} \mid m \geq 0; 0 \leq i_1 < i_2 < \dots < i_m; \text{ and } 0 \leq k_\nu \leq d-1 \text{ for } \nu = 1, \dots, m\}$ is a basis of the vector K -space $L = K\langle b \rangle$, while for every $r \in \mathbf{N}$, its subset $B_r = \{\alpha^{i_1}(b)^{k_1} \dots \alpha^{i_m}(b)^{k_m} \mid m \geq 0; r \leq i_1 < i_2 < \dots < i_m; \text{ and } 0 \leq k_\nu \leq d-1 \text{ for } \nu = 1, \dots, m\}$ is a basis of the vector K -space $K\langle \alpha^r(b) \rangle$.

If $a \in \bigcap_{n=0}^{\infty} K\langle \alpha^n(S) \rangle$ then $a \in K\langle b \rangle$ hence $a = \lambda_1 b_1 + \dots + \lambda_p b_p$ for some $\lambda_1, \dots, \lambda_p \in K$ and $b_1, \dots, b_p \in B$. Let $\alpha^s(b)$ be the highest transform of b that appears in b_1, \dots, b_p . Since $a \in K\langle \alpha^{s+1}(b) \rangle$, a can be written as $a = \mu_1 b'_1 + \dots + \mu_q b'_q$ for some $\mu_1, \dots, \mu_q \in K$ and $b'_1, \dots, b'_q \in B_{s+1}$. We arrive at the equality $\lambda_1 b_1 + \dots + \lambda_p b_p = \mu_1 b'_1 + \dots + \mu_q b'_q$ that can hold only if the both expressions in its sides belong to K . Therefore, $a \in K$, so that $\bigcap_{n=0}^{\infty} K\langle \alpha^n(S) \rangle = K$.

□

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