

Diffspec

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Abstract

The language of schemes, which has proven to be of value to algebraic geometry, has not yet been widely accepted into differential algebraic geometry. One reason may be that there are some “challenges”. In this talk we examine some of those challenges.

The first work on diffspec of a differential ring is due to Keigher and this is the definition we use. It is different from those of Carra’ Ferro and Buium. We first we examine the ring of global sections, which in general is not isomorphic to the given differential ring. This brings up the notions of differential zero, differential unit and AAD (Annihilators Are Differential).

Morphisms of diffspec also pose challenges; they are not always induced from homomorphisms of differential rings. Products and closed subschemes present further challenges.

We also introduce the associated space of constants of a differential scheme and the notion of split differential scheme. These ideas are used in differential Galois theory.

We end with a challenge to the audience to develop theories of quasi-coherent sheaves, group schemes, dimension, singularities, etc.

1 Introduction

Differential algebraic geometry started with Ritt on the “manifolds” of solutions of differential equations. Kolchin modernized it using the language of Weil. For an excellent account and bibliography see Buium and Cassidy [7].

In [17] Kolchin broke from the Weil tradition by axiomatizing the notion of differential algebraic group. This followed closely his axiomatization of the notion of algebraic group which is found in [16]. This was an elegant tour de force, but it has not become widely accepted.

The language of schemes was introduced in the work of Keigher [9, 10, 11, 12, 13] and was continued by Carra’ Ferro [4, 5] and Buium [2].

I was drawn into studying differential schemes because of differential Galois theory. In [14] and [15] Kolchin developed this using the language of Weil. The Galois group is then birationally isomorphic to an algebraic group, but not canonically. This is because the proof used Weil’s “group chunks”. In his book [16] Kolchin used the axiomatics instead. This makes the Galois group canonically an algebraic group, but at the expense of a non-standard definition of algebraic group. Unfortunately neither approach is widely used today.

As it turns out, the use of `diffspec` allows for a canonical identification of the Galois group with a group scheme.

2 Differential rings

References are Kaplansky [8] and Kolchin [16].

Rings are always commutative with identity. A differential ring is a ring \mathcal{R} together with a fixed set of commuting derivations $\Delta = \{\delta_1, \dots, \delta_m\}$. The subring of constants is \mathcal{R}^Δ .

Δ -ring, Δ -ideal, etc., means differential ring, differential ideal, etc.

If S is a subset of \mathcal{R} , then $[S]$ is the smallest Δ -ideal containing S .

Definition 1. \mathcal{R} is a *Keigher ring* if for every Δ -ideal \mathfrak{a} , $\sqrt{\mathfrak{a}}$ is a Δ -ideal.

Every Ritt algebra (algebra over \mathbb{Q}) is a Keigher ring, also any ring of constants. So Keigher rings include all non-differential rings. This is important as we want our theory to be a generalization of both differential algebraic geometry and schemes (ala Grothendieck).

Example 2. $\mathbb{Z}[x]$ where $\delta x = x' = 1$ is not a Keigher ring. (p, x^p) is a Δ -ideal since $(x^p)' = px^{p-1} \in (p, x^p)$, but $\sqrt{(p, x^p)} = (p, x)$ is not.

Were we to use Hasse-Schmidt derivations (Okugawa [21, 22]) every Δ -ring would be a Keigher Δ -ring.

For the sake of this lecture we assume our rings are Keigher rings. This is not essential for most of the what we say, but does make things easier. We fix a Keigher ring \mathcal{R} .

3 Differential schemes

Definition 3. $X = \text{diffspec } \mathcal{R}$ is the set of all prime Δ -ideals of \mathcal{R} . If S is a subset of \mathcal{R} , then $V(S)$ is the set of $\mathfrak{p} \in \text{diffspec } \mathcal{R}$ with $S \subset \mathfrak{p}$. For $b \in \mathcal{R}$, $D(b)$ is the set of $\mathfrak{p} \in \text{diffspec } \mathcal{R}$ with $b \notin \mathfrak{p}$.

Proposition 4. *The sets $V(S)$ are the closed sets in a topology on X .*

Definition 5. The topology is called the *Kolchin topology*.

X is a subset of $\text{spec } \mathcal{R}$ and the Kolchin topology is the subspace topology of the Zariski topology. $D(b)$ form a basis of open sets.

Example 6. X can be empty if \mathcal{R} is not Keigher! Let

$$\mathcal{R} = \mathbb{Z}[x]/(p, x^p) \quad \text{where } x' = 1.$$

Any prime Δ -ideal \mathfrak{p} would contain the image of x and therefore 1, which is impossible.

We define the structure sheaf in the usual way.

Definition 7. For each open set U of X let $\mathcal{O}_X(U)$ be the set of functions $s : U \rightarrow \coprod_{\mathfrak{p} \in U} \mathcal{R}_{\mathfrak{p}}$ satisfying:

1. $s(\mathfrak{p}) \in \mathcal{R}_{\mathfrak{p}}$, and
2. there is an open cover U_i of U and $a_i, b_i \in \mathcal{R}$, such that for each $\mathfrak{p} \in U_i$, $b_i \notin \mathfrak{p}$ and $s(\mathfrak{p}) = a_i/b_i \in \mathcal{R}_{\mathfrak{p}}$.

Basically s is locally a quotient of elements of \mathcal{R} .

Proposition 8. For $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is Δ -isomorphic to $\mathcal{R}_{\mathfrak{p}}$.

Proposition 9. If $\phi : \mathcal{S} \rightarrow \mathcal{R}$ is a Δ -homomorphism, then there is an induced morphism of schemes $({}^a\phi, \phi^\#) : \text{diffspec } \mathcal{R} \rightarrow \text{diffspec } \mathcal{S}$.

4 Global sections

Definition 10. Denote by $\widehat{\mathcal{R}}$ the ring of global sections $\mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X)$ and the canonical mapping by

$$\begin{aligned} \iota = \iota_{\mathcal{R}} : \mathcal{R} &\rightarrow \widehat{\mathcal{R}} \\ \iota(r)(\mathfrak{p}) &= \frac{r}{1} \in \mathcal{R}_{\mathfrak{p}} \quad (\mathfrak{p} \in X). \end{aligned}$$

In the algebraic theory ι is an isomorphism. In our case it is neither injective nor surjective. This is a *major* challenge.

Example 11. Let $\mathcal{R} = \mathbb{Q}[x]$, where $\delta x = x' = 1$. (0) is the only prime Δ -ideal so

$$\widehat{\mathcal{R}} = \mathbb{Q}[x]_{(0)} = \mathbb{Q}(x)$$

and $\iota : \mathbb{Q}[x] \rightarrow \mathbb{Q}(x)$ is injective but not surjective.

Example 12. Let $\mathcal{R} = \mathbb{Q}[x]\{\eta\} = \mathbb{Q}[x]\{y\}/[xy]$ with $x' = 1$. Then $x\eta = 0$.

We claim that $\mathfrak{p} = [\eta]$ is the only prime Δ -ideal in \mathcal{R} . First observe that \mathfrak{p} is a maximal Δ -ideal since $\mathcal{R}/[\eta] \approx \mathbb{Q}[x]$, which is Δ -simple (has no non-zero proper Δ -ideal). Also

$$0 = \eta(x\eta)' = \eta(\eta + x\eta') = \eta^2.$$

Hence any prime Δ -ideal in X contains \mathfrak{p} which proves the claim. So $X = \{\mathfrak{p}\}$ and $\widehat{\mathcal{R}} = \mathcal{R}_{\mathfrak{p}}$. But

$$\iota(\eta) = \frac{\eta}{1} = 0 \in \mathcal{R}_{\mathfrak{p}}$$

since $x\eta = 0$ and $x \notin \mathfrak{p}$. Therefore ι is not injective.

For any $P/Q \in \mathcal{R}_{\mathfrak{p}}$ we can find $e \in \mathbb{N}$ with

$$\frac{P}{Q} = \frac{x^e P}{x^e Q}$$

and $x^e Q$ and $x^e P$ do not involve η, η', \dots . In other words,

$$\widehat{\mathcal{R}} = \mathcal{R}_{\mathfrak{p}} = \mathbb{Q}(x).$$

So ι is not surjective either.

Carra' Ferro [6] addresses this by changing the structure sheaf. Buium [3] changes everything. For him a differential scheme is a scheme (e.g. $\text{spec } \mathcal{R}$) whose sheaf consists of Δ -rings. This is what Umemura [23] calls a “scheme with differentiation”. Keigher [9, 10, 11, 12, 13] has kept the faith.

Proposition 13. *The canonical mapping $\iota_{\widehat{\mathcal{R}}} : \widehat{\mathcal{R}} \rightarrow \widehat{\widehat{\mathcal{R}}}$ is an isomorphism.*

This makes $\widehat{\mathcal{R}}$ a kind of “closure” of \mathcal{R} .

5 Differential zeros

Observe: $r \in \mathcal{R}$ is 0 if and only if $1 \in \text{ann}(r)$.

Definition 14. $r \in R$ is a Δ -zero if $1 \in [\text{ann}(r)]$. The set of Δ -zeros of \mathcal{R} is denoted by $\mathfrak{Z}(\mathcal{R})$.

As the definition indicates, $\text{ann}(r)$ is not necessarily a Δ -ideal. Indeed, if $a \in \text{ann}(r)$, so $ar = 0$, then

$$0 = \delta(ar) = \delta a r + a \delta r.$$

We can multiply by r :

$$0 = \delta a r^2 + ar \delta r = \delta a r^2$$

but now what? (If \mathcal{R} is reduced then we would get $\delta a r = 0$ but in general we cannot say anything.)

Proposition 15. *The kernel of $\iota : \mathcal{R} \rightarrow \widehat{\mathcal{R}}$ is $\mathfrak{Z}(\mathcal{R})$.*

Proof. $\iota(r) = 0$ means that for every $\mathfrak{p} \in X$

$$\iota(r)(\mathfrak{p}) = \frac{r}{1} = 0 \in \mathcal{R}_{\mathfrak{p}}.$$

So there exists $a \notin \mathfrak{p}$ with $ar = 0 \in \mathcal{R}$. Thus $\text{ann}(r)$ is not contained in any prime Δ -ideal so $1 \in [\text{ann}(r)]$ and $r \in \mathfrak{Z}(\mathcal{R})$. \square

Proposition 16. *$\mathfrak{Z}(\mathcal{R})$ is a Δ -ideal of \mathcal{R} .*

Proposition 17. *$\mathfrak{Z}(\mathcal{R}/\mathfrak{Z}(\mathcal{R})) = 0$. $\mathfrak{Z}(\mathcal{R})$ is the smallest Δ -ideal with that property.*

Proposition 18. *Every element of $\mathfrak{Z}(\mathcal{R})$ is nilpotent.*

So every prime Δ -ideal of \mathcal{R} contains $\mathfrak{Z}(\mathcal{R})$. If \mathcal{R} is reduced, $\mathfrak{Z}(\mathcal{R}) = (0)$.

Proposition 19. *$X = \text{diffspec } \mathcal{R}$ and $\text{diffspec}(\mathcal{R}/\mathfrak{Z}(\mathcal{R}))$ are isomorphic.*

Example 20. Let $\mathcal{R} = \mathbb{Q}[x]\{\eta\}$ with $x\eta = 0$. Then $x \in \text{ann}(\eta)$ so $\eta \in \mathfrak{Z}(\mathcal{R})$. Since $\mathcal{R}/[\eta] \approx \mathbb{Q}[x]$ is Δ -simple, $\mathfrak{Z}(\mathcal{R}) = [\eta]$ and is a maximal Δ -ideal.

$$\text{diffspec } \mathcal{R} \approx \text{diffspec } \mathcal{R}/[\eta] \approx \text{diffspec } \mathbb{Q}[x] \approx \text{diffspec } \mathbb{Q}(x).$$

6 AAD rings

Definition 21. \mathcal{R} is AAD (Annihilators Are Differential) if $\text{ann}(r)$ is a Δ -ideal for every $r \in \mathcal{R}$.

In other words, \mathcal{R} is AAD if $ar = 0$ implies $\delta_i ar = 0$ for all $\delta_i \in \Delta$.

Proposition 22. *If \mathcal{R} is AAD then $\mathfrak{Z}(\mathcal{R}) = 0$.*

In particular $\iota : \mathcal{R} \rightarrow \widehat{\mathcal{R}}$ is injective. In this case we usually identify \mathcal{R} with a subring of $\widehat{\mathcal{R}}$.

Proposition 23. *If \mathcal{R} is reduced then \mathcal{R} is AAD. If the derivations on \mathcal{R} are trivial then \mathcal{R} is AAD.*

This is why AAD is important; it includes the rings of classical Δ -algebraic geometry as well as all the rings of algebra.

Proposition 24. *\mathcal{R} is AAD if and only if $\mathcal{R}_{\mathfrak{p}}$ is AAD for every $\mathfrak{p} \in X$.*

In other words, AAD is a local condition. Thus we may define a Δ -scheme to be AAD if all of the stalks are AAD.

Proposition 25. *If \mathcal{R} is any Δ -ring then there is a smallest Δ -ideal $\mathfrak{A}(\mathcal{R})$ such that $\mathcal{R}/\mathfrak{A}(\mathcal{R})$ is AAD.*

Thus we have the analog of a radical. Just as we can reduce any scheme, we can “AAD” any Δ -scheme. And AAD-uction is a nop (no operation - i.e. does nothing) for (algebraic) schemes.

Proposition 26. *$\mathcal{R} \rightarrow \mathcal{R}/\mathfrak{A}(\mathcal{R})$ induces a homeomorphism of $\text{diffspec}(\mathcal{R}/\mathfrak{A}(\mathcal{R}))$ onto X .*

But not an isomorphism of schemes. But neither does $\mathcal{R} \rightarrow \mathcal{R}/\text{nilrad}(\mathcal{R})$.

Proposition 27. *If \mathcal{R} is AAD then $X \approx \widehat{X} = \text{diffspec } \widehat{\mathcal{R}}$.*

This generalizes the theorem

$$\text{spec } R \approx \text{spec } \Gamma(\text{spec } R, \mathcal{O}_{\text{spec } R})$$

of algebraic geometry.

7 Denominators

We suppose that \mathcal{R} is AAD.

A global section can always be written in the form

$$s(\mathfrak{p}) = \begin{cases} \frac{a_1}{b_1} & \text{for } \mathfrak{p} \in D(b_1) \\ \vdots & \\ \frac{a_r}{b_r} & \text{for } \mathfrak{p} \in D(b_r) \end{cases}$$

where $1 \in [b_1, \dots, b_r]$.

If $\mathfrak{p} \in D(b_i) \cap D(b_j) = D(b_i b_j)$ then

$$\frac{a_i}{b_i} = \frac{a_j}{b_j} \in \mathcal{R}_{\mathfrak{p}}$$

so there exists $x_{ij} \notin \mathfrak{p}$ with

$$x_{ij}(a_i b_j - a_j b_i) = 0 \in \mathcal{R}.$$

In particular, $\text{ann}(a_i b_j - a_j b_i) \not\subset \mathfrak{p}$. Therefore

$$b_i b_j \in \sqrt{[\text{ann}(a_i b_j - a_j b_i)]},$$

i.e.

$$(b_i b_j)^e \in [\text{ann}(a_i b_j - a_j b_i)] = \text{ann}(a_i b_j - a_j b_i)$$

for some $e \in \mathbb{N}$. Replace a_i by $a_i b_i^e$ and b_i by b_i^{e+1} and we have

$$a_i b_j = a_j b_i.$$

Therefore for any $\mathfrak{q} \in X$, say $q \in D(b_j)$,

$$b_i s(\mathfrak{q}) - a_i = \frac{b_i a_j}{b_j} - \frac{a_i}{1} = 0 \in \mathcal{R}_{\mathfrak{q}}$$

i.e.

$$b_i s = a_i.$$

Proposition 28. *Suppose that \mathcal{R} is AAD and $s \in \widehat{\mathcal{R}}$. Then there exist $a_1, b_1, \dots, a_r, b_r \in \mathcal{R}$ with $1 \in [b_1, \dots, b_r]$ and*

$$b_i s(\mathbf{q}) = a_i$$

for all $\mathbf{q} \in X$ and all $i = 1, \dots, r$.

This is not quite a “common denominator” since b need not be invertible in $\widehat{\mathcal{R}}$; in fact it could even be a zero divisor.

If we knew that $1 \in (b_1, \dots, b_r)$ then we would have

$$1 \cdot s = \left(\sum_{i=1}^r c_i b_i \right) s = \sum_{i=1}^r c_i (b_i s) = \sum_{i=1}^r c_i a_i \in \mathcal{R}.$$

But that is too much to ask for.

8 Epi

The following is “almost” surjectivity.

Proposition 29. *Suppose that \mathcal{R} is AAD. Then so is $\widehat{\mathcal{R}}$ and*

$$\iota : \mathcal{R} \rightarrow \widehat{\mathcal{R}}$$

is epi in the category of AAD rings.

This means that if \mathcal{S} is an AAD ring and

$$f, g : \widehat{\mathcal{R}} \rightarrow \mathcal{S}$$

satisfy

$$f \circ \iota = g \circ \iota : \mathcal{R} \rightarrow \mathcal{S}$$

then $f = g$.

9 Morphisms

Let $X = \text{diffspec } \mathcal{R}$ and $Y = \text{diffspec } \mathcal{S}$. If $\phi : \mathcal{R} \rightarrow \mathcal{S}$ then there is an induced morphism

$$({}^a\phi, \phi^\#) : (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X).$$

Conversely if $(f, f^\#) : (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$ then we have the picture

$$\begin{array}{ccc} \widehat{\mathcal{R}} & \xrightarrow{f^\#(X)} & \widehat{\mathcal{S}} \\ \uparrow \iota_{\mathcal{R}} & & \uparrow \iota_{\mathcal{S}} \\ \mathcal{R} & & \mathcal{S} \end{array}$$

Definition 30. $f : Y \rightarrow X$ is said to be *realizable* if you can fill in the bottom arrow for some choice of \mathcal{R} and \mathcal{S} with $X \approx \text{diffspec } \mathcal{R}$ and $Y \approx \text{diffspec } \mathcal{S}$.

Actually this is not the complete definition - we should also allow open covers of X and Y .

For example, $f : \text{diffspec } \mathbb{Q}[x] \rightarrow \text{diffspec } \mathbb{Q}(x)$ but there is no homomorphism $\mathbb{Q}(x) \rightarrow \mathbb{Q}[x]$.

On the other hand $\text{diffspec } \mathbb{Q}[x] \approx \text{diffspec } \mathbb{Q}(x)$ and there is a homomorphism $\mathbb{Q}(x) \rightarrow \mathbb{Q}(x)$. So f is realizable.

Proposition 31. *If X and Y are AAD then any $f : Y \rightarrow X$ is realizable.*

This is because $X \approx \text{diffspec } \widehat{\mathcal{R}}$. If X and Y are not AAD, I don't know. I don't have a proof or a counterexample.

Here's a related question. If $f : Y \rightarrow X$ is given we get $f^\#(X) : \widehat{\mathcal{R}} \rightarrow \widehat{\mathcal{S}}$. Conversely, given $\phi : \widehat{\mathcal{R}} \rightarrow \widehat{\mathcal{S}}$ we can get $f : Y \rightarrow X$, and we can show that

$$\phi \circ \iota = f^\#(X) \circ \iota.$$

Can we conclude that $f^\#(X) = \phi$? If \mathcal{R} is AAD then yes, because ι is epi. In general I do not know.

10 Products

Suppose that $X \rightarrow A$ and $Y \rightarrow A$ are morphisms of affine Δ -schemes. We would like to form the product

$$X \times_A Y.$$

Suppose $A = \text{diffspec } \mathcal{A}$, then this should be easy:

$$X \times_A Y = \text{diffspec } (\mathcal{R} \otimes_{\mathcal{A}} \mathcal{S}).$$

However \mathcal{R} and \mathcal{S} are not necessarily \mathcal{A} algebras! So the tensor product does not make sense.

If everything is AAD then we can form

$$X \times_A Y = \text{diffspec } (\widehat{\mathcal{R}} \otimes_{\widehat{\mathcal{A}}} \widehat{\mathcal{S}})$$

which gives an answer, but one that is particularly easy to work with.

It follows from the general theory that if \mathcal{R} and \mathcal{S} are \mathcal{A} -algebras then

$$\text{diffspec}(\mathcal{R} \otimes_{\mathcal{A}} \mathcal{S}) \approx \text{diffspec}(\widehat{\mathcal{R}} \otimes_{\widehat{\mathcal{A}}} \widehat{\mathcal{S}})$$

which is not a priori obvious.

11 Closed subschemes

This material is from [20] where I assumed “reduced” to make the proofs easier. I suspect that everything goes through for AAD.

Definition 32. By a *closed subscheme* Y of X is meant a scheme (Y, \mathcal{O}_Y) , where $Y \subset X$, together with a closed immersion $(Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$.

Proposition 33. *If $\mathfrak{a} \subset \mathcal{R}$ is a Δ -ideal then $V(\mathfrak{a}) \approx \text{diffspec}(\mathcal{R}/\mathfrak{a})$ is a closed subscheme of X .*

However there may be other closed subscheme structures on the closed set $V(\mathfrak{a})$. They come from Δ -ideals \mathfrak{b} with $V(\mathfrak{a}) = V(\mathfrak{b})$.

In algebraic geometry this proposition has a converse, every closed subscheme comes from an ideal. However all we can prove the following ([20, Proposition 22.6]).

Proposition 34. *Suppose that \mathcal{R} is reduced and Y is a reduced closed subscheme of X . Then there is a radical Δ -ideal $\mathfrak{a} \subset \mathcal{R}$ such that Y is isomorphic to $\text{diffspec}(\mathcal{R}/\mathfrak{a})$.*

Since we do not have the theory of quasi-coherent sheaves, our proof does not follow the usual one of algebraic geometry. I believe that it is true with AAD replacing “reduced”, using a different proof. In the more general case I have no idea. In fact I cannot even prove that a closed subscheme of X must be affine.

12 Space of constants

Definition 35. Define a sheaf \mathcal{O}_X^Δ on X by the formula

$$\mathcal{O}_X^\Delta(U) = \mathcal{O}_X(U)^\Delta$$

where $U \subset X$ is open.

Proposition 36. *If $\mathfrak{p} \in X$ then the stalk $\mathcal{O}_{X,\mathfrak{p}}^\Delta$ is isomorphic to $\mathcal{R}_\mathfrak{p}^\Delta$ and is a local ring.*

However $\mathcal{R}_\mathfrak{p}^\Delta$ is not of the form $(\mathcal{R}^\Delta)_{\mathfrak{p}_o}$ for some $\mathfrak{p}_o \subset \mathcal{R}^\Delta$.

Definition 37. We denote the local ringed space $(X, \mathcal{O}_X^\Delta)$ by X^Δ . It is called the *local ringed space of constants of X* .

Note that the topological spaces of X and X^Δ are the same, it is only the sheaves that are different.

Proposition 38. *If \mathcal{R} is reduced then $\widehat{\mathcal{R}}^\Delta = \mathcal{R}^\Delta = \widehat{\mathcal{R}^\Delta}$.*

In general X^Δ is simply a local ringed space, but in certain fortuitous cases it is actually a scheme.

If it is an affine scheme then $X^\Delta = \text{spec } \mathcal{R}^\Delta$. The prototypical example of this is

$$\mathcal{R} = \mathcal{F}[\mathcal{R}^\Delta] = \mathcal{F} \otimes_{\mathcal{F}^\Delta} \mathcal{R}^\Delta$$

where \mathcal{F} is a Δ -field.

13 Split differential schemes

Fix a Δ -field \mathcal{F} of characteristic 0 with field of constants \mathcal{C} . If $Y = \text{spec } \mathcal{D}$ is a scheme over \mathcal{C} we make it into a Δ -scheme trivially, i.e. we make \mathcal{D} into a Δ -ring of constants. Evidently $Y = \text{spec } \mathcal{D} = \text{diffspec } \mathcal{D}$.

Next form the product of Δ -schemes:

$$\mathcal{F} \times_{\mathcal{C}} Y = \text{diffspec } \mathcal{F} \times_{\text{diffspec } \mathcal{C}} Y = \text{diffspec}(\mathcal{F} \otimes_{\mathcal{C}} \mathcal{D}).$$

This construction generalizes to not necessarily affine schemes.

Definition 39. $X = \text{diffspec } \mathcal{R}$ is *split* if there is a scheme Y such that $X \approx \mathcal{F} \times_{\mathcal{C}} Y$.

Buium in [3] has a similar construction except that he takes the product in the category of schemes, not Δ -schemes.

Proposition 40. *If X is reduced and split then $X \approx \mathcal{F} \times_{\mathcal{C}} X^\Delta$.*

Proposition 41. *If X is reduced, split and X^Δ is affine, then for every $\mathfrak{p} \in X$,*

$$\mathcal{R}_{\mathfrak{p}} = \mathcal{F}[\mathcal{R}^\Delta]_{\mathfrak{p}} \approx (\mathcal{F} \otimes_{\mathcal{C}} \mathcal{R}^\Delta)_{\mathfrak{p}}.$$

But we cannot conclude that $\mathcal{R} = \mathcal{F}[\mathcal{R}^\Delta] \approx \mathcal{F} \otimes_{\mathcal{C}} \mathcal{R}^\Delta$.

If X is split but X^Δ is not affine we can say very little. And that is the case that occurs for Δ -Galois theory.

14 Differential Galois theory

The reason I became interested in Δ -schemes is for differential Galois theory. For details see [20].

Suppose that $\mathcal{F} \subset \mathcal{G}$ are Δ -fields with \mathcal{G} strongly normal over \mathcal{F} (i.e. Δ -Galois). Form the Δ -ring

$$\mathcal{P} = \mathcal{G} \otimes_{\mathcal{F}} \mathcal{G}.$$

One can identify Δ -automorphisms of \mathcal{G} over \mathcal{F} , i.e. the Galois group, with maximal Δ -ideals of \mathcal{P} , i.e. with the closed points of

$$P = \text{diffspec } \mathcal{P}.$$

Proposition 42. *P^Δ is a scheme.*

But not necessarily affine. In fact it is affine if and only if \mathcal{G} is a Picard-Vessiot extension of \mathcal{F} .

Proposition 43. *P is split, i.e. $P \approx \mathcal{G} \times_e P^\Delta$.*

Moreover there is a natural coring structure on \mathcal{P} . The comultiplication is

$$\begin{aligned} \mu : \mathcal{P} &\rightarrow \mathcal{P} \otimes_{\mathcal{G}} \mathcal{P} \\ a \otimes_{\mathcal{F}} b &\mapsto a \otimes_{\mathcal{F}} 1 \otimes_{\mathcal{G}} 1 \otimes_{\mathcal{F}} b. \end{aligned}$$

This does not make \mathcal{P} into a Hopf algebra, it does not make P into a Δ -group scheme (it we knew what that meant!). But it does induce a multiplication on X^Δ and does make X^Δ into a group scheme.

Thus $P = \mathcal{G} \times_e P^\Delta$ has a structure of Δ -group scheme by virtue of being a base extension. That would make $\widehat{\mathcal{P}}$ into a Hopf algebra, but the relation between the comultiplication in $\widehat{\mathcal{P}}$ and the μ above is not at all clear. Someone needs to sort this out. Volunteers?

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