Solutions of linear ordinary differential equations in terms of special functions

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$$4(x-1)^8 \frac{d^2y}{dx^2} = (3 - 50x + 61x^2 - 60x^3 + 45x^4 - 18x^5 + 3x^6)y(x)$$

has the general solution

$$y(x) = C_1(x-1)^{3/2} Ai\left(\frac{x(x-2)}{(x-1)^2}\right) + C_2(x-1)^{3/2} Bi\left(\frac{x(x-2)}{(x-1)^2}\right)$$

Applications

This is where the project proposal's vision falls somewhat short of what it could be. This is not really due to a shortcoming of the scientific aspirations of the project, but from a fundamental gap between what mathematicians currently call "closed form" and what people interested in applications refer to as "closed form". To put it succinctly, the Liouvillian notion of closed form does not correspond to a practitioner's definition of the same term. More precisely, an expression like $\int \exp\left(\frac{1}{x^2}\right)\ln(x)dx$ is considered to be a closed form in the Liouvillian sense but $J_\nu(z)$ (a Bessel function) is not. A typical engineer would reverse those statements. The Liouvillian ideas make for a beautiful and powerful mathematical theory, but does not give a satisfactory practical treatment of finding closed form expressions for solutions of differential equations. Although much of the research proposed in CAFE are in fact applicable without much extra work to such solutions, the focus is not there, and we believe it should be.

What are special functions?

A *special function* is a non-Liouvillian solution $F(x)$ of

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = 0 \qquad \text{with } a_0, a_1 \in C(x)$$

$$y'' - xy = 0 \rightarrow \text{ Airy functions: } Ai(x), Bi(x)$$

$$y'' + \frac{1}{x}y' - \left(\frac{\nu^2}{x^2} \pm 1\right)y = 0 \rightarrow \text{ Bessel fns: } I_\nu(x), J_\nu(x), K_\nu(x), Y_\nu(x)$$

$$y'' - \left(\frac{1}{4} - \frac{\mu}{x} - \frac{1/4 - \nu^2}{x^2}\right)y = 0 \rightarrow \text{ Whittaker fns: } W_{\mu,\nu}(x), M_{\mu,\nu}(x)$$

- Differential Galois theory: the Airy, Bessel and Whittaker equations have Galois group $SL_2(\mathbb{C})$, as well as other equations not admitting closed–form special function solutions.

- Equivalence problem: let $F_1(x), F_2(x)$ be a (known) fundamental solution set of $y'' + a_1 y' + a_0 y = 0$. Looking for solutions of the form $m(x)F_i(\xi(x))$ of another equation $y'' + b_1 y' + b_0 y = 0$ is the same as looking for a point transformation

$$x \to \xi(x) \quad y \to m(x)y$$

that transforms $y'' + a_1 y' + a_0 y = 0$ into $y'' + b_1 y' + b_0 y = 0$.
This problem always has a solution (for LODE of order 2)...
...but not always in closed-form!

Closed-form equivalence problem

Given $y'' + a_1 y' + a_0 y = 0$ and $y'' + b_1 y' + b_0 y = 0$, find whether there exist closed–form $m(x)$ and $\xi(x)$ such that

$$x \to \xi(x) \quad y \to m(x)y$$

transforms $y'' + a_1 y' + a_0 y = 0$ into $y'' + b_1 y' + b_0 y = 0$. Such transforms do not form a group under composition anymore.

When the equation to solve is of the form $y'' - vy = 0$ we get

$$m\xi' \neq 0\,, \qquad \frac{m'}{m} = \frac{1}{2}\left(a_1(\xi)\xi' - \frac{\xi''}{\xi'}\right)$$

and

$$3\xi''^2 - 2\xi'\xi''' + \left(a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi)\right)\xi'^4 - 4v\xi'^2 = 0$$

Rational solutions of

$$3\xi''^2 - 2\xi'\xi''' + \Big(a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi)\Big)\xi'^4 - 4v\xi'^2 = 0$$

For $P, Q \in C[x]$, define $\nu_\infty(P/Q) = \deg(q) - \deg(p)$.
Let $v = P/Q \in C(x)$ and $Q = Q_1 Q_2^2 \cdots Q_e^e$ be a squarefree factorisation of its denominator.

If $\nu_\infty(a_1^2 + 2a_1' - 4a_0) < 2$, then any solution $\xi \in C(x)$ can be written as $\xi = A/D$ where

$$D = \prod_i Q^i_{(2-\nu_\infty(a_1^2+2a_1'-4a_0))i+2}$$

and $A \in C[x]$ is such that either $\deg(A) \leq \deg(D) + 1$ or

$$\deg(A) = \deg(D) + \frac{2 - \nu_\infty(v)}{2 - \nu_\infty(a_1^2 + 2a_1' - 4a_0)}$$

Algebraic solutions for ξ

$$\xi = A\left(x^{1/(2-\nu_\infty(a_1^2+2a_1'-4a_0))}\right)\prod_{i>2} Q_i^{(i-2)/(2-\nu_\infty(a_1^2+2a_1'-4a_0))}$$

Solving $y'' = xy$ in terms of the Bessel equation

$$y'' + \frac{1}{x}y' - \left(\frac{\nu^2}{x^2}+1\right) = 0$$

$a_1 = 1/x$, $a_0 = -(1+\nu^2/x^2)$, $a_1^2 + 2a_1' - 4a_0 = (4\nu^2-1)/x^2 + 4$.

$$\xi = c_0 + c_1 x^{\frac{1}{2}} + c_2 x + c_3 x^{\frac{3}{2}}$$

$$\nu = \pm\frac{1}{3} \qquad c_0 = c_1 = c_2 = 0 \quad c_3 = \pm\frac{2}{3}$$

$$C_1\sqrt{x}I_{1/3}\left(\frac{2}{3}x\sqrt{x}\right) + C_2\sqrt{x}K_{1/3}\left(\frac{2}{3}x\sqrt{x}\right)$$

Open questions

- Is solving a nonlinear system really necessary?
- How about higher order equations?
- Are there necessary conditions for specific special functions?