The Differential Galois Group of a Laplace Integral
KSDA Lecture

Alberto Baider

March 16, 2002

Abstract

The "directional" Laplace transform $u = \mathcal{L}_d(r)$ of a rational function r satisfies a linear homogeneous differential equation $Lu = 0$ over the field $\mathbb{C}\{z\}$ of germs of meromorphic functions at $0 \in \mathbb{C}$. This talk is meant to be an illustration of a theorem of Martinet and Ramis describing the Galois group of the Picard-Vessiot extension associated with the operator L. The basic ingredients in that theorem are the "Stokes matrices", the "exponential torus", and the "formal monodromy". While the exponential torus and the formal monodromy are in principle computable objects, the Stokes matrices are essentially transcendental, notoriously difficult to calculate, and known in only a few cases of classical equations. After a description of all these objects, I will calculate them explicitly in our situation.

Preliminary Background

A linear system of n differential equations

$$(*) \qquad \partial_z u = Au, \quad \partial_z = \tfrac{d}{dz}$$

with coefficients in $K = \mathbb{C}\{z\}$, the field of germs of meromorphic functions at $0 \in \mathbb{C}$, always admits a formal fundamental solution of the form

$$\widehat{X} = \widehat{F} z^L e^{\mathrm{diag}(q_1, \cdots, q_n)}$$

with $\widehat{F} \in \widehat{K} := \mathbb{C}((z))$, L a constant coefficients matrix, and $q_i \in z^{-\frac{1}{\nu}}\mathbb{C}[z^{-\frac{1}{\nu}}]$ for some positive integer ν in the range $1, \cdots, n$. If 0 is a regular singular point (Fuchsian case) all the q_i vanish and $\widehat{F}$ is actually a convergent power series. We are here interested in the irregular singular case, which is characterized by the appearance of at least one non-trivial exponential e^{q_i}.

Let $\mathbf{P} = K\!<\!\widehat{X}\!>$ (resp. $\widehat{\mathbf{P}} = \widehat{K}\!<\!\widehat{X}\!>$) be the Picard-Vessiot extension(s) of our system, the second one the result of enlarging the base field from K to $\widehat{K}$. To describe $Gal(\mathbf{P}|K)$ one begins with the easier task of describing the subgroup $Gal(\widehat{\mathbf{P}}|\widehat{K})$. Noting that the matrix z^L appearing in $\widehat{X}$ is defined as $e^{\log(z)L}$, we see that $\widehat{\mathbf{P}}$ is a subfield of

$$\widehat{K} < z^{\frac{1}{\nu}}, z^{\lambda_1}, \cdots, z^{\lambda_n}, \log(z), e^{q_1}, \cdots, e^{q_n} >$$

where the $\lambda' s$ are the eigenvalues of L and where $\log(z)$ is only needed when L is not semisimple.

The exponential torus $\mathcal{T}$ is the subgroup of $Gal(\widehat{\mathbf{P}}|\widehat{K})$ that fixes the subfield

$$\widehat{K}\!<\!z^{\frac{1}{\nu}}, z^{\lambda_1}, \cdots, z^{\lambda_n}, \log(z)\!> .$$

Since every element $\sigma \in \mathcal{T}$ must send each e^{q_i} to some multiple $\chi_\sigma(q_i)e^{q_i}$ of itself, it is not hard to verify that σ defines a $\mathbb{C}_*$-valued character χ_σ on the free group $\mathcal{F}$ generated by the q_i in $z^{-\frac{1}{\nu}}\mathbb{C}[z^{-\frac{1}{\nu}}]$ and that the correspondence $\sigma \mapsto \chi_\sigma$ is an isomorphism between $\mathcal{T}$ and the group of characters of $\mathcal{F}$. Since the group of characters of $\mathbb{Z}$ is $\mathbb{C}_*$, we see that $\mathcal{T}$ is indeed isomorphic to a torus, namely $\mathbb{C}_*^r$, where r is the rank of $\mathcal{F}$.

The formal monodromy is the map γ resulting from analytic continuation of $z, \log(z)$, and the e^{q_i} along a simple loop around the origin, and fixing $\widehat{K}$.

It is worthwhile noting that the exponentials e^{q_i}, coming as they are from a meromorphic matrix A, while not necessarily fixed, are permuted under analytic continuation around the origin.

Theorem A. The group generated by the exponential torus $\mathcal{T}$ and the formal monodromy γ is Zariski dense in $Gal(\widehat{\mathbf{P}}|\widehat{K})$.

To state a similar characterization of $Gal(\mathbf{P}|K)$ we need some definitions.

- By **angular sector of opening s** about the direction $d = \mathbb{R}_+ e^{i\theta}$ and of radius r we mean a set of the form

$$V = \{z \in \mathbb{C} \mid 0 < |z| < r,\ |\arg(z) - \theta| < s\}.$$

- **Asymptotic expansion:** A holomorphic function f on an angular sector V is said to have an asymptotic expansion $\hat{f} = \sum_0^\infty a_j z^j$ if for every sector W with $\overline{W} \setminus \{0\} \subset V$ there exists a sequence of positive constants C_n such that

$$|f(z) - \sum_0^{n-1} a_j z^j| \leq C_n |z|^n$$

for $z \in W$.

The formal power series $\hat{f}$ is uniquely determined by f whenever it exists, and the map sending f to $\hat{f}$ is a differential ring homomorphism J between the space of functions having asymptotic expansions on V and $\mathbb{C}[[z]]$.

- An asymptotic expansion is said to be k-Gevrey if the C_n can be taken to be of the form $C^n (n!)^{\frac{1}{k}}$ or, equivalently by Stirling's formula, of the form $D^n n^{\frac{n}{k}}$.

- **k-summability:** A formal power series $\hat{f}$ is said to be k-summable in the direction d, if there is a function f k-Gevrey asymptotic to $\hat{f}$ on an open sector about d of **opening wider than** π/k. It is a classical result (Watson's lemma) that if such f exists, it is unique, and the map $S_{k,d} : \hat{f} \mapsto f$, a right inverse to J, is also a differential ring homomorphism defined on the space of formal power series that are k-summable in the direction d.

- **Multisummability:** Let $\mathbf{L}$ be a finite set of positive reals. A formal power series $\hat{f}$ is said to be $\mathbf{L}$-multisummable in the direction d if $\hat{f} = \sum_{k \in \mathbf{L}} \hat{f}_k$, where $\hat{f}_k$ is k-summable in the direction d. Although the $\hat{f}_k$ are not unique, the sum $S_d(\hat{f}) = \sum S_{k,d}(\hat{f}_k)$ is well defined on a sector of opening $\frac{\pi}{\max\{\mathbf{L}\}}$ about d, and S_d is again a differential ring right inverse to J on the appropriate space. We say that $\hat{f}$ is $\mathbf{L}$-multisummable if it is $\mathbf{L}$-multisummable in every but finitely many directions.

- A **level** of the system (*) is one the positive numbers $k \in \dfrac{\mathbb{Z}}{\nu}$ that occur as fractionary degrees in z^{-1} of the *differences*

$$q_i - q_j \in z^{-\frac{1}{\nu}}\mathbb{C}[z^{-\frac{1}{\nu}}].$$

  We denote by $\mathbf{L}$ the (finite) set of levels.

- By **singular direction** we mean a ray d of maximal decay of one of the exponentials

$$e^{(q_i-q_j)(z)} = e^{\frac{a}{z^k}(1+o(1))}, \text{as } z \longrightarrow 0,$$

  where $o(1) \in z^{\frac{1}{\nu}}\mathbb{C}[z^{\frac{1}{\nu}}]$. There are finitely many such directions and they are precisely those rays along which $\Re e(\frac{a}{z^k}) < 0$. We will also call them **Stokes directions**, although this nomenclature is not uniform in the literature. Some authors prefer to call these directions *anti-Stokes* reserving "Stokes" for those rays across which an exponential undergoes a change from flat to explosive.

Example The exponential $e^{-\frac{a}{z}}$ has only one Stokes direction: $d = \mathbb{R}_+ a$. If H_d denotes the open half plane bisected by d, the exponential is 1-Gevrey flat there, but explodes exponentially on the opposite half plane $-H_d$. Note that Watson's lemma prevents a non-zero function from being 1-Gevrey asymptotic to zero on an angular sector containing $\overline{H_d} \setminus \{0\}$.

Theorem B. (Turritin, Hukuhara, Balser, Jurkat, Malgrange Sibuya, Ramis...)Let $s = \frac{\pi}{\max(\mathbf{L})}$. We have:

1. $\widehat{F}$ is $\mathbf{L}$ multisummable and for every non-singular direction

$$X_d := S_d(\widehat{F}) z^L e^{\mathrm{diag}(q_1, \cdots, q_n)}$$

is a holomorphic fundamental solution of (*) on a sector V_d of opening s.

2. Let $\mathbf{V}$ be the n-dimensional space of formal solutions of (*), and $\mathbf{V_d}$ its holomorphic counterpart on V_d. The correspondence $\widehat{X} \mapsto X$ induced by multisummation defines a **Galois** isomorphism $\mathcal{S}_d : \mathbf{V} \to \mathbf{V}_d$.

3. Orient the circle say counterclockwise, and let $d_- < d_+$ be two "nearby" directions such that $|\arg(d_+) - \arg(d_-)| < s$. Then

$$X_{d_-}|_{V_{d_-} \cap V_{d_+}} = X_{d_+}|_{V_{d_-} \cap V_{d_+}} C$$

for some constant matrix C (the Stokes matrix of d), which is the identity unless the pair d_-, d_+ encloses a Stokes direction d. If so, the map $St_d : \mathbf{V} \to \mathbf{V}$ (the Stokes map of d) determined by $\widehat{X} \mapsto \widehat{X}C$ is **Galois** and independent of the pair d_-, d_+ enclosing the Stokes direction d. In fact

$$St_d = (\mathcal{S}_{d_-})^{-1} \circ AC \circ \mathcal{S}_{d_+},$$

where AC is analytic continuation from V_{d_-} to V_{d_+}.

Theorem C. (Martinet, Ramis) The group generated by the exponential torus $\mathcal{T}$, the formal monodromy γ, and the Stokes maps St_d is Zariski dense in $Gal(\mathbf{P}|K)$.

Laplace Integrals

Let r be a rational function and let $\mathcal{P}$ be the set of its poles which we assume not to contain 0. We call a direction **polar** if it is of the form $\mathbb{R}_+ a$ where $a \in \mathcal{P}$. Consider the function $u = u_d$ defined by the integral

$$\mathcal{L}_d(r)(z) = \int_d e^{-\frac{t}{z}} r(t) dt$$

where d is any *non-polar* direction. The integral *converges uniformly* on compact subsets of the open half plane H_d bisected by d, where $e^{-\frac{t}{z}}$ decays exponentially as $t \to \infty$, so that u_d is holomorphic there. Whereas even differentiation under the integral sign is legal on H_d, the integral diverges on the opposite open half plane $-H_d$ on which $e^{-\frac{t}{z}}$ explodes exponentially as $t \to \infty$. Thus u_d is a family of holomorphic functions parameterized by the space of directions $S^1 \setminus \mathcal{P}$, each defined on a different open half plane H_d.

Example: $\mathcal{L}_d(t^n)(z) = n! z^{n+1}$. Thus $\mathcal{L}_d$ maps a polynomial to a polynomial divisible by z. Note that in this case u_d does not depend on d.

A differential equation for $\mathbf{u}_d$

The formula $\partial_z(e^{-\frac{t}{z}}) = \frac{t}{z^2} e^{-\frac{t}{z}}$ says that $e^{-\frac{t}{z}}$ is an eigenfunction of the derivation $D_z := z^2 \partial_z$ with eigenvalue t. Thus $D_z \mathcal{L}_d(r) = \mathcal{L}_d(tr)$ or more generally $p(D_z) \circ \mathcal{L}_d(r) = \mathcal{L}_d(pr)$ for any polynomial p. In particular, if $r = \frac{q}{p}$ then $p(D_z) u_d = \mathcal{L}_d(q)$ is a polynomial divisible by z, so that all the u_d satisfy the *same* linear differential equation

$$(**) \qquad L(u) = 0$$

with

$$L = (\partial_z)^{\deg(q)+1} \circ \frac{1}{z} \circ p(D_z).$$

To tighten this up, and for simplicity, we now assume that $r = \frac{1}{p}$, where p has only simple zeros $a_1, \cdots, a_n$, no two of which belong to the same ray. In this case

$$L = \partial_z \circ \frac{1}{z} \circ \prod_{i=1}^{n} (D_z - a_i)$$

A basis of solutions

Since $e^{-\frac{a_i}{z}}$ is in the kernel of $D_z - a_i$ and these operators commute with one another, it follows that on any half plane H_d bisected by d, all the $e^{-\frac{a_i}{z}}$ are in the kernel of L besides u_d. Inspection of L shows that these functions are independent, and since L has order $n+1$, we conclude that

$$\{e^{-\frac{a_1}{z}}, \cdots, e^{-\frac{a_n}{z}}, u_d\}$$

is a fundamental system of solutions of (**) on H_d.

Dependence on directions

Theorem: Let $d_- < d_+$ be two "nearby" directions and let V_{d_-,d_+} be the open infinite sector bounded by them. Then

$$u_{d_+} - u_{d_-} = -2\pi i \sum_{a_j \in V_{d_-,d_+}} r_j e^{-\frac{a_j}{z}}$$

where r_j is the residue of r at the pole a_j. In particular if there are no poles between d_- and d_+, u_{d_-} coincides with u_{d_+} on $H_{d_-} \cap H_{d_+}$, so they analytically continue each other to $H_{d_-} \cup H_{d_+}$.
Proof. This is just Cauchy's theorem, together with the fact that the integral on the "infinite" arc of the sector V_{d_-,d_+} is zero.
If a_-, a, a_+ are any three "consecutive" poles, the theorem says that u_d is constant as a function of d for d between the poles, thus defining two functions near $\mathbb{R}_+ a$ that we will call u_{a_-}, u_{a_+} holomorphic on $H_{a_-} \cup H_a, H_a \cup H_{a_+}$ respectively, the "jump" at $\mathbb{R}_+ a$ being given by $u_{a_+} - u_{a_-} = -2\pi i r_a e^{-\frac{a}{z}}$.

Asymptotic expansion

Consider the Taylor series $\sum c_j t^j$ of $r(t)$ and let

$$\hat{u} = \widehat{\mathcal{L}}(\sum c_j t^j) = \sum j! c_j z^{j+1}$$

be its **formal** Laplace integral obtained by
termwise integration. The series so obtained is clearly *divergent*, yet it plays a central role in what follows, as the next result indicates.

Theorem. We have:

1. $\hat{u}$ is 1-summable in every non-polar direction d with sum u_d.
2. $\{e^{-\frac{a_1}{z}}, \cdots, e^{-\frac{a_n}{z}}, \hat{u}\}$ is a formal fundamental solution to $L(u) = 0$.
3. The only non-trivial Stokes directions are the polar directions $d_j := \mathbb{R}_+ a_j$.
4. For every non-polar direction d,

$$\mathcal{S}_d(\{e^{-\frac{a_1}{z}}, \cdots, e^{-\frac{a_n}{z}}, \hat{u}\}) = \{e^{-\frac{a_1}{z}}, \cdots, e^{-\frac{a_n}{z}}, u_d\}.$$

5. The Stokes matrix of the polar direction d_j is:

$$St_{d_j} = \begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & & 1 & \cdots & -2\pi i\, r_j \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & & 1 \end{pmatrix}$$

where r_j is the residue of r at a_j.

It follows from 5. that the Zariski closure of the group generated by the Stokes matrices is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & & \mathbb{C} \\ 0 & 1 & 0 & \cdots & & \mathbb{C} \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & & 1 & \cdots & \mathbb{C} \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & & 1 \end{pmatrix}$$

The exponential torus in our situation is the group

$$\begin{pmatrix} \mathbb{C}_* & 0 & 0 & \cdots & & 0 \\ 0 & \mathbb{C}_* & 0 & \cdots & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & & \mathbb{C}_* & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & & 1 \end{pmatrix}.$$

Since the formal monodromy is trivial in our case we finally have:

Theorem. The Galois group of the Picard-Vessiot extension generated by the Laplace integral(s) $\mathcal{L}_d(r)$ over K is of the form

$$\begin{pmatrix} \mathbb{C}_* & 0 & 0 & \cdots & & \mathbb{C} \\ 0 & \mathbb{C}_* & 0 & \cdots & & \mathbb{C} \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & & \mathbb{C}_* & \cdots & \mathbb{C} \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & & 1 \end{pmatrix} \approx (\mathbb{C} \ltimes \mathbb{C}_*)^n$$

with $\mathbb{C} \ltimes \mathbb{C}_*$ the semidirect product defined by $(a, \lambda)(b, \mu) = (a + \lambda b, \lambda\mu)$.

References

1. J.Martinet and J.P. Ramis, *Elementary Acceleration and Multisummability*, Annales de L'Institut Henri Poincaré , Physique Théorétique 54, (1991), 331-401.

2. M. van der Put, *Differential Galois Theory, Universal Rings and Universal Groups*, to appear in the Proceedings of the International Workshop on Differential Algebra, Rutgers University at Newark, November 2-3, 2000.

3. M. van der Put and M, Singer, *Differential Galois Theory*, book in preparation.

Department of Mathematics and Statistics, Hunter College of the City University of New York, New York, NY 10021

alberto.baider@hunter.cuny.edu