INVARIANTS OF LINEAR q-DIFFERENCE EQUATIONS IN THE COMPLEX DOMAIN AND GALOIS THEORY

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Abstract

In the two lectures, we will present a complete theory of (algebraic and transcendental) invariants of linear q-difference equations in the complex domain. This theory was initiated by G. D. Birkhoff (in particular in his famous 1913 paper about various generalizations of the Riemann-Hilbert problem). Near the end of his life, G.D. Birkhoff proposed (at the end of a joint paper with P.E. Guenther) a large program in this direction (1941):

Up to the present time, the theory of linear q-difference equations has lagged noticeably behind the sister theories of linear difference and differential equations. In the opinion of the authors, the use of the canonical system, as formulated above in a special case, is destined to carry the theory of q-difference equations to a comparable degree of completeness. This program includes in particular the complete theory of convergence and divergence of formal series, the explicit determination of the essential transcendental invariants (constants in the canonical form), the inverse Riemann theory both for the neighborhood of $x = \infty$ and in the complete plane (case of rational coefficients), explicit integral representation of the solutions, and finally the definition of q-sigma periodic matrices, so far defined essentially only in the case $n = 1$. Because of its extensiveness this material cannot be presented here.

This program was completed quite recently by J. Sauloy, C. Zhang and J.P. Ramis.

First Lecture. The Fuchsian Case

We will begin with the Fuchsian (or regular singular) case. We will first recall briefly the invariant theory for the differential regular singular case (Riemann-Hilbert theory). After we will develop the parallel q-difference case. The classical monodromy representation is more or less replaced by a Birkhoff connection matrix. In our new approach this matrix is elliptic.

If we perform a continuous limit ($q \to 1$), q-difference equations tend to differential equations and there is a beautiful confluence relation between the discrete and the continuous theories of invariants.

In the linear complex differential regular case there is a nice relation between the invariant theory and the differential Galois theory: "the" differential Galois group "is" the Zariski closure of the image of the monodromy representation. We will present a similar result in the q-difference case. The first step (in the regular case) is due to P. Etingof. The general regular singular case is a lot more delicate; a clear presentation needs the use of groupoids and flat fiber spaces on elliptic curves.

Finally, for the regular singular case, the situation is quite nice and we have more or less a complete theory: invariants, confluence, Galois theory, and their relations. Moreover for the important family of q-analogs of hypergeometric equations (basic hypergeometric equations), it is possible to compute explicitely everything.

In the irregular case, even if we made big progresses, we did not reached the same degree of achievement...

Second Lecture. The Irregular Case

We will first recall briefly the invariant theory in the irregular differential case. The divergence of formal power series solutions play a central role. There exists a delicate theory of "canonical" resummations of such series and the ambiguities in the resummations give rise to the Stokes phenomena which give new analytic invariants (a new sort of monodromy). Summability theory is strongly related to some Gevrey estimates.

In the q-difference case the situation is in some sense less and more difficult.

Less difficult because surprisingly there exists locally at zero and ∞ a canonical *analytic* filtration (related to a Newton polygon). In the differential case this filtration is only *formal*.

More difficult because the q-analog of k-summability is quite delicate. An important fact is that it does not exist a natural process for choosing a q-analog. In particular, if one wants a q-analog of the Laplace transform (which is needed for k-summability theory), there is some choices:

– Kernel choices: we must choose among three quite natural q-analog of the exponential function.

– Contour choices: we must choose a continuous contour (continuous spiral) or a discrete contour (discrete spiral).

These choices are strongly related to a choice of q-constant fields. We developed systematically the summation theory in all these directions, but our aim was to get at the end elliptic matrices as resummations ambiguities (q-Stokes phenomena). We succeeded using good choices for the kernels (theta functions) and the contours (discrete spirals). At the end we built a complete theory of invariants along these lines. We will present shortly three aspects: algebraic, geometric and analytic (q-Gevrey estimates and q-analog of multisummability). This finish Birkhoff program.

Now, what about continuous limit process (confluence) and Galois theory.

We just began the work in these directions. We can guess on some examples that there will be a limit process from the q-Stokes phenomena towards the Stokes phenomena. But we have only a conjecture (involving cohomology spaces).

The relation between the invariant theory and the Galois theory remains delicate in the irregular case. The main difficulty of the Fuchsian case (in general the natural fiber functors are not compatible with tensor products) is even more serious now. This is strongly related with classification of fiber bundles on elliptic curves and geometric non abelian class field theory...